Indecomposable involutive 2-permutational solutions of the Yang–Baxter equation

Přemysl Jedlička
with Agata Pilitowska

Department of Mathematics
Faculty of Engineering (former Technical Faculty)
Czech University of Life Sciences (former Czech University of Agriculture) in Prague

Stará Lesná
6th September 2023

Faculty of Engineering

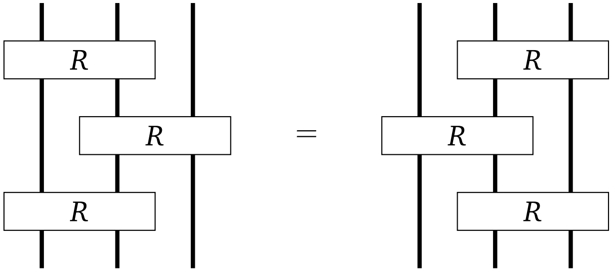
CZECH UNIVERSITY OF LIFE SCIENCES PRAGUE

Yang–Baxter equation

Definition

Let V be a vector space. A homomorphism $R : V \otimes V \to V \otimes V$ is called a *solution of Yang–Baxter equation* if it satisfies

$$(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R).$$



Set-theoretic solutions

Definition

Let X be a set. A mapping $r : X \times X \to X \times X$ is called a *set-theoretic solution of Yang–Baxter equation* if it satisfies

$$(r \times \mathrm{id}_X)(\mathrm{id}_X \times r)(r \times \mathrm{id}_X) = (\mathrm{id}_X \times r)(r \times \mathrm{id}_X)(\mathrm{id}_X \times r).$$

A solution $r : (x,y) \mapsto (\sigma_x(y), \tau_y(x))$ is called *non-degenerate* if σ_x and τ_y are bijections, for all $x,y \in X$. A solution is called *involutive* if $r^2 = \mathrm{id}_{X^2}$.

Observation

If r is involutive then $\tau_y(x) = \sigma^{-1}_{\sigma_x(y)}(x)$.

Set-theoretic solutions

Definition

Let X be a set. A mapping $r : X \times X \to X \times X$ is called a *set-theoretic solution of Yang–Baxter equation* if it satisfies

$$(r \times \mathrm{id}_X)(\mathrm{id}_X \times r)(r \times \mathrm{id}_X) = (\mathrm{id}_X \times r)(r \times \mathrm{id}_X)(\mathrm{id}_X \times r).$$

A solution $r : (x,y) \mapsto (\sigma_x(y), \tau_y(x))$ is called *non-degenerate* if σ_x and τ_y are bijections, for all $x,y \in X$. A solution is called *involutive* if $r^2 = \mathrm{id}_{X^2}$.

Observation

If r is involutive then $\tau_y(x) = \sigma^{-1}_{\sigma_x(y)}(x)$.

Equational variety

Proposition

Involutive solutions form a variety with signature $(X,\sigma,\tau,\sigma^{-1},\tau^{-1})$ *and axioms*

$$\sigma_x^{-1}\sigma_x(y)=y \qquad \tau_x^{-1}\tau_x(y)=y$$
$$\sigma_x\sigma_x^{-1}(y)=y \qquad \tau_x\tau_x^{-1}(y)=y$$
$$\sigma_x\sigma_y(z)=\sigma_{\sigma_x(y)}\sigma_{\tau_y(x)}(z) \qquad \tau_y(x)=\sigma^{-1}_{\sigma_x(y)}(x)$$

Definition

An involutive solution X is called 2-permutational if, for all $x,x',y\in X$,

$$\sigma_{\sigma_x(y)}=\sigma_{\sigma_{x'}(y)}.$$

Equational variety

Proposition

Involutive solutions form a variety with signature $(X,\sigma,\tau,\sigma^{-1},\tau^{-1})$ *and axioms*

$$\sigma_x^{-1}\sigma_x(y)=y \qquad \tau_x^{-1}\tau_x(y)=y$$
$$\sigma_x\sigma_x^{-1}(y)=y \qquad \tau_x\tau_x^{-1}(y)=y$$
$$\sigma_x\sigma_y(z)=\sigma_{\sigma_x(y)}\sigma_{\tau_y(x)}(z) \qquad \tau_y(x)=\sigma^{-1}_{\sigma_x(y)}(x)$$

Definition

An involutive solution X is called 2-permutational if, for all $x,x',y\in X$,

$$\sigma_{\sigma_x(y)}=\sigma_{\sigma_{x'}(y)}.$$

Permutation group

Definition

Let (X, σ, τ) be an involutive solution. The group

$$\mathcal{G}(X) = \langle \sigma_x \mid x \in X \rangle$$

is called the *permutation group* of X or the *involutive Yang-Baxter group* of X.

Definition

We say that an involutive solution is *indecomposable* if $\mathcal{G}(X)$ acts transitively on X.

Permutation group

Definition

Let (X, σ, τ) be an involutive solution. The group

$$\mathcal{G}(X) = \langle \sigma_x \mid x \in X \rangle$$

is called the *permutation group* of X or the *involutive Yang-Baxter group* of X.

Definition

We say that an involutive solution is *indecomposable* if $\mathcal{G}(X)$ acts transitively on X.

Solutions of size pq

Theorem (M. Castelli, G. Pinto, W. Rump)

Let (X, σ, τ) be an indecomposable involutive solution of size pq, where p, q are primes, such that $\mathcal{G}(X)$ is abelian. Then X is 2-permutational.

There is only one such solution, up to isomorphism if $p \neq q$, and there are $p + 1$ such solutions if $p = q$.

Solutions of size pq

Theorem (M. Castelli, G. Pinto, W. Rump)

Let (X, σ, τ) be an indecomposable involutive solution of size pq, where p, q are primes, such that $\mathcal{G}(X)$ is abelian. Then X is 2-permutational.
There is only one such solution, up to isomorphism if $p \neq q$, and there are $p + 1$ such solutions if $p = q$.

Displacement group

Definition

Let (X, σ, τ) be an involutive solution. Then *displacement* group or the *transvection* group of X is the group

$$\mathrm{Dis}(X) = \langle \sigma_x \sigma_y^{-1} \mid x, y \in X \rangle.$$

Theorem (W. Rump)

$\mathrm{Dis}(X)$ *is a normal subgroup of* $\mathcal{G}(X)$ *and* $\mathcal{G}(X) = \mathrm{Dis}(X)\langle \sigma_x \rangle$, *for any* $x \in X$.

Theorem (W. Rump; P. J., A. P.)

Let (X, σ, τ) *be an involutive 2-permutational solution. Then* $\mathrm{Dis}(X)$ *is abelian.*

Displacement group

Definition

Let (X, σ, τ) be an involutive solution. Then *displacement* group or the *transvection* group of X is the group

$$\mathrm{Dis}(X) = \langle \sigma_x \sigma_y^{-1} \mid x, y \in X \rangle.$$

Theorem (W. Rump)

$\mathrm{Dis}(X)$ is a normal subgroup of $\mathcal{G}(X)$ and $\mathcal{G}(X) = \mathrm{Dis}(X)\langle \sigma_x \rangle$, for any $x \in X$.

Theorem (W. Rump; P. J., A. P.)

Let (X, σ, τ) be an involutive 2-permutational solution. Then $\mathrm{Dis}(X)$ is abelian.

Displacement group

Definition

Let (X, σ, τ) be an involutive solution. Then *displacement* group or the *transvection* group of X is the group

$$\mathrm{Dis}(X) = \langle \sigma_x \sigma_y^{-1} \mid x, y \in X \rangle.$$

Theorem (W. Rump)

$\mathrm{Dis}(X)$ *is a normal subgroup of* $\mathcal{G}(X)$ *and* $\mathcal{G}(X) = \mathrm{Dis}(X)\langle \sigma_x \rangle$, *for any* $x \in X$.

Theorem (W. Rump; P. J., A. P.)

Let (X, σ, τ) *be an involutive 2-permutational solution. Then* $\mathrm{Dis}(X)$ *is abelian.*

Example on groups

Example

Let $X = \{1, 2, 3, 4, 5\}$ and let

σ	1	2	3	4	5
1	2	1	5	4	3
2	2	1	3	5	4
3	2	1	4	3	5
4	2	1	4	3	5
5	2	1	4	3	5

Then

$$\mathcal{G}(X) = \{\mathrm{id}_X, (1,2)(3,5), (1,2)(4,5), (1,2)(3,4), (3,4,5), (5,4,3)\}$$

and

$$\mathrm{Dis}(X) = \{\mathrm{id}_X, (3,4,5), (5,4,3)\}.$$

Indecomp. 2-permut. solutions with abelian group

Proposition (P. J., A. P., A. Zamojska-Dzienio)

Let (X, σ, τ) be an idecomposable 2-permutational involutive solution with $\mathcal{G}(X)$ abelian. Then

- $\mathrm{Dis}(X)$ *is cyclic,*
- $\mathcal{G}(X)$ *has* 2 *generators,*
- $o(\sigma_x) = o(\sigma_y)$, *for all* $x, y \in X$.

Theorem (P. J., A. P., A. Zamojska-Dzienio)

For finite solutions, there are 3 parameters of isomorphism, namely $n_1, n_2, r \in \mathbb{Z}$, such that

$$n_1 \mid n_2, \qquad 0 \leqslant r < n_2/n_1, \qquad n_2 \mid n_1 r^2.$$

Then $|X| = n_1 \cdot n_2$ *and* $\mathcal{G}(X) \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$.

Indecomp. 2-permut. solutions with abelian group

Proposition (P. J., A. P., A. Zamojska-Dzienio)

Let (X, σ, τ) be an idecomposable 2-permutational involutive solution with $\mathcal{G}(X)$ abelian. Then

- $\mathrm{Dis}(X)$ *is cyclic,*
- $\mathcal{G}(X)$ *has* 2 *generators,*
- $o(\sigma_x) = o(\sigma_y)$*, for all* $x, y \in X$.

Theorem (P. J., A. P., A. Zamojska-Dzienio)

For finite solutions, there are 3 parameters of isomorphism, namely $n_1, n_2, r \in \mathbb{Z}$*, such that*

$$n_1 \mid n_2, \qquad 0 \leqslant r < n_2/n_1, \qquad n_2 \mid n_1 r^2.$$

Then $|X| = n_1 \cdot n_2$ *and* $\mathcal{G}(X) \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$.

Generators of the displacement group

Proposition (P. J., A. P.)

Let (X, σ, τ) *be an indecomposable involutive* 2*-permutational solution. Choose* $e \in X$ *and let* $d = \sigma_e(e)$*. Then* $o(\sigma_e) = o(\sigma_d)$ *and*

$$\mathcal{G}(X) = \langle \sigma_e, \sigma_d \rangle \qquad \text{and} \qquad \mathrm{Dis}(X) = \left\langle \sigma_e^{-i} \sigma_d \sigma_e^{i-1} \mid i \in \mathbb{Z} \right\rangle.$$

Indecomposable solutions with non-abelian permutation group

Theorem (P. J., A. P.)

There exists an indecomposable solution that homomorphically maps onto any indecomposable involutive 2-permutational solution.

Idea of the proof.

$$\begin{array}{lll} \mathbb{Z} & \ldots & \text{free cyclic group} \\ \bigoplus_{\mathbb{Z}} \mathbb{Z} & \ldots & \text{free abelian group with } \omega \text{ generators} \end{array}$$

$(\bigoplus_{\mathbb{Z}} \mathbb{Z}) \rtimes \mathbb{Z}$ maps onto $\mathcal{G}(X) = \mathrm{Dis}(X)\langle\sigma_x\rangle$ □

Indecomposable solutions with non-abelian permutation group

Theorem (P. J., A. P.)

There exists an indecomposable solution that homomorphically maps onto any indecomposable involutive 2-permutational solution.

Idea of the proof.

$\mathbb{Z}$ … free cyclic group

$\bigoplus_{\mathbb{Z}} \mathbb{Z}$ … free abelian group with ω generators

$(\bigoplus_{\mathbb{Z}} \mathbb{Z}) \times \mathbb{Z}$ maps onto $\mathcal{G}(X) = \mathrm{Dis}(X)\langle \sigma_x \rangle$ □

Indecomposable solutions with non-abelian permutation group

Theorem (P. J., A. P.)

There exists an indecomposable solution that homomorphically maps onto any indecomposable involutive 2-permutational solution.

Idea of the proof.

$\mathbb{Z}$ ... free cyclic group

$\bigoplus_{\mathbb{Z}} \mathbb{Z}$ ... free abelian group with ω generators

$(\bigoplus_{\mathbb{Z}} \mathbb{Z}) \rtimes \mathbb{Z}$ maps onto $\mathcal{G}(X) = \mathrm{Dis}(X)\langle\sigma_x\rangle$ □

Indecomposable solutions with non-abelian permutation group

Theorem (P. J., A. P.)

There exists an indecomposable solution that homomorphically maps onto any indecomposable involutive 2-permutational solution.

Idea of the proof.

$$\begin{array}{ccl} \mathbb{Z} & \ldots & \text{free cyclic group} \\ \bigoplus_{\mathbb{Z}} \mathbb{Z} & \ldots & \text{free abelian group with } \omega \text{ generators} \end{array}$$

$(\bigoplus_{\mathbb{Z}} \mathbb{Z}) \rtimes \mathbb{Z}$ maps onto $\mathcal{G}(X) = \mathrm{Dis}(X)\langle \sigma_x \rangle$ □

Constructing all the indecom. inv. 2-perm. solut.

Theorem (P. J., A. P.)

A complete set of invariants for a finite indecomposable involutive 2-permutational solution are

- $m, n \in \mathbb{N}$;
- *an abelian group* A *of size* n *with less than* m *generators;*
- *an element* $r \in A$;
- H, *a subgroup of* $\mathbb{Z}^{m-1}$, *such that* $\mathbb{Z}^{m-1}/H \cong A$.

The solution then constructed has $m \cdot n$ *elements and its displacement group is isomorphic to* A.

Corollary

Let $s \in \mathbb{N}$. *Then there are at least* $2^{k/2} - 1$ *indecomposable solutions of size* $k = 2^s$.

Constructing all the indecom. inv. 2-perm. solut.

Theorem (P. J., A. P.)

A complete set of invariants for a finite indecomposable involutive 2-permutational solution are

- $m, n \in \mathbb{N}$;
- *an abelian group* A *of size* n *with less than* m *generators;*
- *an element* $r \in A$;
- H, *a subgroup of* $\mathbb{Z}^{m-1}$, *such that* $\mathbb{Z}^{m-1}/H \cong A$.

The solution then constructed has $m \cdot n$ *elements and its displacement group is isomorphic to* A.

Corollary

Let $s \in \mathbb{N}$. *Then there are at least* $2^{k/2} - 1$ *indecomposable solutions of size* $k = 2^s$.

Numbers of indecomposable involutive solutions

n	1	2	3	4	5	6	7	8
solutions	1	2	5	23	88	595	3456	34530
2-perm.	1	2	5	19	70	359	2095	16332
indecom.	1	1	1	5	1	10	1	100
ind. 2-perm.	1	1	1	3	1	10	1	19
ind. 2-perm. abel. $\mathcal{G}$	1	1	1	3	1	1	1	3
ind. 2-perm. cycl. $\mathcal{G}$	1	1	1	2	1	1	1	2

n	9	10	11	12	13	14	15	16
sol.	321931	4895272						
ind.	16	36	1		1			
i. 2-p.	13	36	1	136	1	134	151	403
i. 2-p. a.	4	1	1	3	1	1	1	7
i. 2-p. c.	3	1	1	2	1	1	1	4