Skew left braces and 2-reductive solutions of the Yang–Baxter equation

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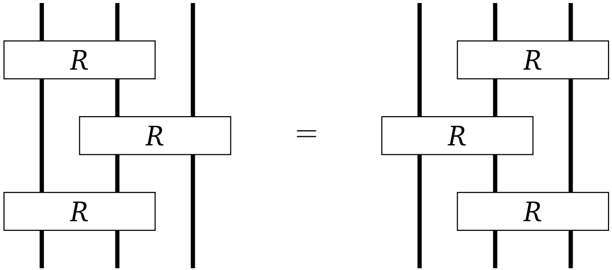
Yang–Baxter equation

Definition

Let V be a vector space. A homomorphism $R: V \otimes V \to V \otimes V$ is called a *solution of Yang–Baxter equation* if it satisfies

$$(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R).$$



Set-theoretic solutions

Definition

Let X be a set. A mapping $r : X \times X \to X \times X$ is called a *set-theoretic solution of Yang–Baxter equation* if it satisfies

$$(r \times \mathrm{id}_X)(\mathrm{id}_X \times r)(r \times \mathrm{id}_X) = (\mathrm{id}_X \times r)(r \times \mathrm{id}_X)(\mathrm{id}_X \times r).$$

A solution $r : (x,y) \mapsto (\sigma_x(y), \tau_y(x))$ is called *bijective* if r is a bijection. It is called *non-degenerate* if σ_x and τ_y are bijections, for all $x,y \in X$.

Involutive solutions

Observation

A structure (X,r) is a solution if and only if σ_x and τ_y are permutations, for all $x,y\in X$, satisfying

$$\sigma_x\sigma_y=\sigma_{\sigma_x(y)}\sigma_{\tau_y(x)}$$
$$\tau_{\sigma_{\tau_y(x)}(z)}\sigma_x(y)=\sigma_{\tau_{\sigma_y(z)}(x)}\tau_z(y)$$
$$\tau_x\tau_y=\tau_{\tau_x(y)}\tau_{\sigma_y(x)}$$

Definition

A solution is called *involutive* if $r^2=\mathrm{id}_{X^2}$.

Observation

If r is involutive then $\tau_y(x)=\sigma^{-1}_{\sigma_x(y)}(x)$.

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Left braces

Definition (W. Rump)

A set B equipped with operations $+$ and $\circ$ is called a *left brace* if

- $(B, +)$ is an abelian group;
- $(B, \circ)$ is a group;
- for all $a, b, c \in B$, we have $a \circ (b + c) = a \circ b + a \circ c - a$.

Example

Let R be a ring and let $n \in J(R)$. Let

$$a \circ b = a + anb + b.$$

Then $(B, +, \circ)$ is a left brace.

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Involutive solutions associated to left braces

Proposition

Let $(B, +, \circ)$ *be a left brace. The mapping* $\lambda : B \to \mathfrak{S}_B$ *defined by*

$$\lambda_a(b) = a \circ b - a$$

is a homomorphism $B \to \mathrm{Aut}(B, +)$.

Proposition

Let $(B, +, \circ)$ *be a left brace. If we define* $r : B^2 \to B^2$ *as*

$$r(a, b) = (\lambda_a(b), \lambda^{-1}_{\lambda_a(b)}(a))$$

then (B, r) *is an involutive solution.*

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Retracts of involutive solutions

Definition

Let (X, σ, τ) be an involutive solution. We define a relation $\sim$ on X

$$x \sim y \text{ if and only if } \sigma_x = \sigma_y.$$

The set $\{[x]_\sim \mid x \in X\}$ with operations

$$\sigma_{[x]_\sim}([y]_\sim) = [\sigma_x(y)]_\sim \qquad \text{and} \qquad \tau_{[y]_\sim}([x]_\sim) = [\tau_y(x)]_\sim$$

is called the *retract* of X and denoted by $\mathrm{Ret}(X)$.

Theorem (P. Etingof, T. Schedler, A. Soloviev)

Let (X, σ, τ) be an involutive solution. Then $\mathrm{Ret}(X)$ is a well-defined involutive solution.

Definition

We say that an involutive solution (X, σ, τ) has *multipermutation level k* if k is the smallest integer such that $|\mathrm{Ret}^k(X)| = 1$.

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Example on multipermutation level

Example

Let R be a commutative ring and let $n \in R$ be a nilpotent element of degree k. Then, when defining

$$a \circ b = a + anb + b,$$

we have

$$\lambda_a(c) = anc + c$$

and

$$a \sim b \text{ if and only if } na = nb.$$

Hence $\operatorname{Ret}(R) \cong nR$ and (R, r) is an involutive solution of multipermutation level k.

Ideals in left braces

Definition

A subset I of a left brace $(B, +, \circ)$ is called an *ideal* if I is a subgroup of $(B, +)$, I is a normal subgroup of $(B, \circ)$ and $\lambda_a(I) \subseteq I$, for each $a \in B$.

Definition

The set

$$\mathrm{Soc}(B) = \{s \in B \mid \forall a \in B \quad s + a = s \circ a\}$$

is an ideal of B called the *socle*.

Observation

$\mathrm{Soc}(B) = \mathrm{Ker}\,\lambda$

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Nilpotency of left braces

Definition

Let $(B, +, \circ)$ be a left brace. We define

- $B_0 = B$,
- $B_{i+1} = B_i/\mathrm{Soc}(B_i)$, for $i \geqslant 0$.

We say that B is *nilpotent* of class k if k is the least integer such that $|B_k| = 1$.

Theorem (W. Rump)

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2-reductive solutions

Definition

We say that an involutive solution is 2-reductive if, for all $x, y, z \in X$, $\sigma_{\sigma_x(y)}(z) = \sigma_y(z)$.

Proposition (T. Gateva-Ivanova)

Let (X, σ, τ) be an involutive solution. Then the following conditions are equivalent:

- *X is 2-reductive,*
- *$\sigma_x \in \mathrm{Aut}(X)$, for each $x \in X$, i.e. $\sigma_x\sigma_y(z) = \sigma_{\sigma_x(y)}\sigma_x(z)$,*
- *X has multip. level at most 2 and, for all $x \in X$, $\tau_x = \sigma_x^{-1}$.*

Corollary

Let (X, σ, τ) be a 2-reductive involutive solution then $\sigma_x\sigma_y = \sigma_y\sigma_x$, for all $x, y \in X$.

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Let (X, σ, τ) be a 2-reductive involutive solution then $\sigma_x\sigma_y = \sigma_y\sigma_x$, for all $x, y \in X$.

Construction of involutive 2-reductive solutions

Theorem (P. J., A. Pilitowska, A. Zamojska-Dzienio)

Let us have

- *an index set* I,
- *abelian groups* A_i, *for* $i \in I$,
- *a matrix of constants* $c_{i,j} \in A_j$, *for* $i, j \in I$.

Then the set $X = \bigsqcup_{i \in I} A_i$ *with operation* $\sigma : X \times X \to X$ *defined by*

$$\sigma_a(b) = b + c_{i,j}, \qquad \text{for } a \in A_i \text{ and } b \in A_j$$

is a 2-reductive involutive solution.
Conversely, every 2-reductive involutive solution can be obtained this way.

Corollary

Each abelian group is isomorphic to the permutation group of a 2-reductive involutive solution.

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Numbers of 2-reductive solutions

n	1	2	3	4	5	6	7	8
involutive solutions	1	2	5	23	88	595	3456	34530
multip. level 2	1	2	5	19	70	359	2095	16332
2-reductive	1	2	5	17	65	323	1960	15421
mp level 2, not 2-red.	0	0	0	2	5	36	135	911

n	9	10	11
2-reductive	155889	2064688	35982357

n	12	13	14
2-reductive	832698007	25731050861	1067863092309

Left braces and 2-reductive solutions

Definition (L. Childs)

A left brace $(B, +, \cdot)$ is called a bi-left brace if

$$a + (b \circ c) = (a + b) \circ a^{-} \circ (a + c),$$

for all $a, b, c \in B$.

Theorem (L. Stefanello, S. Trappeniers)

Let $(B, +, \circ)$ be a left brace. Then its associated solution is 2-reductive if and only if B is a bi-left brace.

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Let (X, σ, τ) be a solution. We define a relation $\sim$ on X as

$$x \sim y \text{ if and only if } \sigma_x = \sigma_y \text{ and } \tau_x = \tau_y.$$

The *retract* and the *multipermutation level* are defined analogously as in the involutive case.

Theorem

Let (X, σ, τ) be a solution. Then $\mathrm{Ret}(X)$ is a well-defined solution.

2019: V. Lebed, L. Vendramin: injective case
2019: P. J., A. Pilitowska, A. Zamojska-Dzienio: general case
2022: F. Cedó, E. Jespers, Ł. Kubat, A. Van Antwerpen, C. Verwimp: shorter proof

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Non-involutive 2-reductive solutions

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Lemma

A 2-reductive solution is of multipermutation level 2 and the group generated by σ_x and τ_x is abelian.

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is a 2-reductive solution.
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Inverse solution

Observation

Let (X, r) be a solution. Then (X, r^{-1}) is also a solution, called the inverse *solution.*

We denote $r^{-1} = (\hat{\sigma}, \hat{\tau})$.

Proposition (P. J., A. Pilitowska)

Let (X, σ, τ) be a 2-reductive solution. Then the inverse solution $(X, \hat{\sigma}, \hat{\tau})$ is 2-reductive as well.

Proof.

Let $\sigma_a(b) = b + c_{i,j}$ and $\tau_b(a) = a + d_{j,i}$.
Then $\hat{\sigma}_a(b) = b - d_{i,j}$ and $\hat{\tau}_b(a) = a - c_{j,i}$. □

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Then $\hat{\sigma}_a(b) = b - d_{i,j}$ and $\hat{\tau}_b(a) = a - c_{j,i}$. $\square$

Skew left braces

Definition (L. Guarnieri, L. Vendramin)

A set B equipped with operations $+$ and $\circ$ is called a *skew left brace* if

- $(B, +)$ is a group;
- $(B, \circ)$ is a group;
- for all $a, b, c \in B$, we have $a \circ (b + c) = a \circ b - a + a \circ c$.

Example

Let G be a group. Then $(G, +, +_{op})$ is a skew left brace.

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Solutions associated to skew left braces

Proposition (L. Guarnieri, L. Vendramin)

Let $(B, +, \circ)$ be a skew left brace. The mapping $\lambda : B \to \mathfrak{S}_B$ defined by $\lambda_a(b) = -a + a \circ b$ is a homomorphism $B \to \mathrm{Aut}(B, +)$.

Proposition (D. Bachiller)

Let $(B, +, \circ)$ be a skew left brace. The mapping $\rho : B \to \mathfrak{S}_B$ defined by $\rho_b(a) = (\lambda_a(b))^{-1} \circ a \circ b$ is an anti-homomorphism, that means $\rho_{a \circ b} = \rho_b \rho_a$.

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Let $(B, +, \circ)$ be a left brace. If we define $r : B^2 \to B^2$ as

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Ideals in skew left braces

Definition

A subset I of a skew left brace $(B, +, \circ)$ is called an *ideal* if I is a normal subgroup of $(B, +)$, I is a normal subgroup of $(B, \circ)$ and $\lambda_a(I) \subseteq I$, for each $a \in B$.

Definition

The set

$$\mathrm{Soc}(B) = \{s \in B \mid a + s = s + a = s \circ a\}$$

is an ideal of B called the *socle*.

Proposition (D. Bachiller)

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Nilpotency of left braces

Definition

Let $(B, +, \circ)$ be a skew left brace. We define

- $B_0 = B$,
- $B_{i+1} = B_i/\mathrm{Soc}(B_i)$, for $i \geqslant 0$.

We say that B is *nilpotent* of class k if k is the least integer such that $|B_k| = 1$.

Theorem (D. Bachiller)

A skew left brace $(B, +, \circ)$ is nilpotent of class k if and only if its associated solution has multipermutation level k

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Opposite skew left braces

Definition (A. Koch, P. J. Truman)

Let $(B, +, \circ)$ be a skew left brace. Then $(B, +_{op}, \circ)$ is a skew left brace called the *opposite* skew left brace.

Theorem (A. Koch, P. J. Truman)

The solution associated to $(B, +_{op}, \circ)$ is inverse to the solution associated to $(B, +, \circ)$.

Corollary

- $\hat{\lambda}_a(b) = (a \circ b) - a,$
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Bi-skew left braces

Definition (L. Childs)

A skew left brace $(B, +, \circ)$ is called a bi-skew left brace if $(B, \circ, +)$ is a skew left brace as well.

Theorem (L. Stefanello, S. Trappeniers)

Let $(B, +, \circ)$ be a skew left brace. Then B is a bi-skew left brace if and only if

$$\lambda_{\hat{\lambda}_a(b)} = \lambda_b,$$

for each $a, b \in B$.

Theorem (A. Caranti)

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Distributive solutions

Theorem (P. J., A. Pilitowska)

Let (X, σ, τ) be a solution. TFAE:

- $\sigma_{\hat{\sigma}_x(y)} = \sigma_y$,
- $\sigma_{\tau_x(y)} = \sigma_y$,
- $\sigma_x\sigma_y = \sigma_{\sigma_x(y)}\sigma_x$,
- $\hat{\tau}_x = \sigma_x^{-1}$,
- $\sigma_x \in \mathrm{Aut}(X)$,

for all $x, y \in X$.

Corollary

Let $(B, +, \circ)$ be a skew left brace. TFAE:

- *B is a bi-skew left brace,*
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Equations of 2-reductivity and skew braces

Proposition (P. J., A. Pilitowska)

Let $(B, +, \circ)$ be a skew left brace. Then

- $\lambda_{\lambda_a(b)} = \lambda_b$ *if and only if* λ *is a homomorphism* $(B, +) \to \mathrm{Aut}(B, \circ)$*, that means* $\lambda_{a+b} = \lambda_a\lambda_b$*;*
- $\lambda_{\rho_a(b)} = \lambda_b$ *if and only if* λ *is an anti-homomorphism* $(B, +) \to \mathrm{Aut}(B, \circ)$*, that means* $\lambda_{a+b} = \lambda_b\lambda_a$*;*
- $\rho_{\rho_a(b)} = \rho_b$ *if and only if* ρ *is a homomorphism* $(B, +) \to \mathfrak{S}_X$*, that means* $\rho_{a+b} = \rho_a\rho_b$*;*
- $\rho_{\lambda_a(b)} = \rho_b$ *if and only if* ρ *is an anti-homomorphism* $(B, +) \to \mathrm{Aut}(B, \circ)$*, that means* $\rho_{a+b} = \rho_b\rho_a$.

Skew left braces and 2-reductivity

Theorem (P. J., A. Pilitowska)

Let $(B, +, \circ)$ be a skew left brace. TFAE

- *the solution (B, λ, ρ) is 2-reductive,*
- $\lambda_{a+b} = \lambda_{b+a} = \lambda_a \lambda_b$ *and* $\rho_{a+b} = \rho_{b+a} = \rho_a \rho_b$,
- (B, λ, ρ) *has multipermutation level at most 2,*
- $(B, +, \circ)$ *is nilpotent of degree at most 2,*
- $(B, +_{op}, \circ)$ *is nilpotent of degree at most 2.*