Distributive biracks and solutions of Yang-Baxter equation

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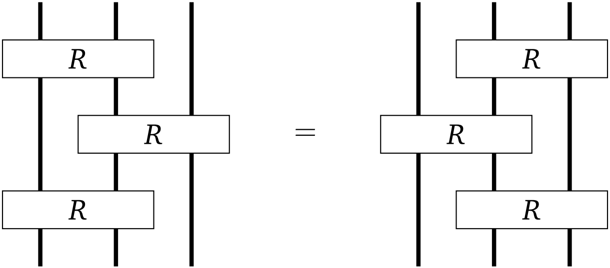
Yang–Baxter equation

Definition

Let V be a vector space. A homomorphism $R : V \otimes V \to V \otimes V$ is called a *solution of Yang–Baxter equation* if it satisfies

$$(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R).$$



Set-theoretic solutions

Definition

Let X be a set. A mapping $r: X \times X \to X \times X$ is called a *set-theoretic solution of Yang–Baxter equation* if it satisfies

$$(r \times \mathrm{id}_X)(\mathrm{id}_X \times r)(r \times \mathrm{id}_X) = (\mathrm{id}_X \times r)(r \times \mathrm{id}_X)(\mathrm{id}_X \times r).$$

A solution $r: (x,y) \mapsto (\sigma_x(y), \tau_y(x))$ is called *non-degenerate* if σ_x and τ_y are bijections, for all $x, y \in X$. A solution r is called *involutive* if $r^2 = \mathrm{id}_{X\times X}$.

Example

Let $(G, \cdot)$ be a group. Then

$$r_1 : (a,b) \mapsto (a^{-1}ba, a)$$

$$r_2 : (a,b) \mapsto (ab^{-1}a^{-1}, ab^2)$$

are both non-degenerate solutions.

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Definition of a birack

Definition

A *birack* is an algebra $(X, \circ, \bullet, \backslash, /)$ that satisfies

$$x\backslash(x \circ y) = y, \qquad (x \bullet y)/y = x,$$
$$x \circ (x\backslash y) = y, \qquad (x/y) \bullet y = x,$$
$$x \circ (y \circ z) = (x \circ y) \circ ((x \bullet y) \circ z),$$
$$(x \circ y) \bullet ((x \bullet y) \circ z) = (x \bullet (y \circ z)) \circ (y \bullet z),$$
$$(x \bullet y) \bullet z = (x \bullet (y \circ z)) \bullet (y \bullet z),$$

Observation

If $(X, \circ, \bullet, \backslash, /)$ is a birack then $(x \circ y, x \bullet y)$ is a non-degenerate solution. Conversely, if (σ_x, τ_y) is a non-degenerate solution then, by setting $x \circ y = \sigma_x(y)$, $x \bullet y = \tau_y(x)$, $x\backslash y = \sigma_x^{-1}(y)$ and $x/y = \tau_y^{-1}(x)$, we obtain a birack.

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Distributive biracks

Definition

Let $(X, \circ, \bullet, \backslash, /)$ be a birack. We say that X is *distributive* if it satisfies

$$x \circ (y \circ z) = (x \circ y) \circ (x \circ z),$$
$$(x \bullet y) \bullet z = (x \bullet z) \bullet (y \bullet z).$$

Example

Let $(X, \circ, \backslash)$ be a left distributive left quasigroup. Then $(X, \circ, \bullet, \backslash, /)$ with $\bullet$ defined by $x \bullet y = x/y = x$, for all $x, y \in X$, is a distributive birack.
Let $(Y, \bullet, /)$ be a right distributive right quasigroup. Then $(Y, \circ, \bullet, \backslash, /)$ with $\circ$ defined by $x \circ y = x \backslash y = y$, for all $x, y \in Y$, is a distributive birack.
Moreover, $X \times Y$ is a distributive birack.

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Multiplication group

Definition

Let $(X, \circ, \bullet, \backslash, /)$ be a birack. We define permutations

$$L_x : a \mapsto x \circ a \qquad \text{and} \qquad \mathbf{R}_x : a \mapsto a \bullet x,$$

for all $x \in X$. We define the *mutliplication group* $\mathrm{Mlt}(X)$ as

$$\mathrm{Mlt}(X) = \langle L_x, \mathbf{R}_x;\ x \in X \rangle.$$

Characterisations of distributive biracks

Theorem (P.J., A.P., A.Z.-D.)

Let X be a birack. Then the following conditions are equivalent:

- *X is distributive,*
- $\mathrm{Mlt}(X) \leqslant \mathrm{Aut}(X)$.

Theorem (P.J., A.P., A.Z.-D.)

Let $(X, \circ, \bullet, \backslash, /)$ be a binary algebra. Then TFAE:

- *X is a distributive birack,*
- *X satisfies the following five properties:*
  - *$(X, \circ, \backslash)$ is a left distributive left quasigroup,*
  - *$(Y, \bullet, /)$ is a right distributive right quasigroup,*
  - *$L_x = L_{x \bullet y}$, for all $x, y \in X$,*
  - *$\mathbf{R}_y = \mathbf{R}_{x \circ y}$, for all $x, y \in X$,*
  - *$L_x \mathbf{R}_y = \mathbf{R}_y L_x$, for all $x, y \in X$.*

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Nilpotency of groups

Definition

Let G be a group. The *lower central series* of G is defined as

- $G_0 = G$,
- $G_i = [G_{i-1}, G]$, for all $i > 0$.

We say that G is *nilpotent of class k* if k is the smallest number such that $|G_k| = 1$. In particular, the trivial group is nilpotent of class 0.

Question:

Does the nilpotency class of $\mathrm{Mlt}(X)$ imply some algebraic property of the birack X?

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Question:

Does the nilpotency class of $\mathrm{Mlt}(X)$ imply some algebraic property of the birack X?

Retraction congruence of biracks

Definition

Let $(X, \circ, \bullet, \backslash, /)$ be a birack. We define a relation $\approx$ on X as follows:

$$x \approx y \text{ if and only if } L_x = L_y \text{ and } \mathbf{R}_x = \mathbf{R}_y.$$

This equivalence is called the *retraction* of X.

Theorem (P. J., A. P., A. Z.-D.)

$\approx$ is a congruence of every birack.

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Multipermutation level

Definition

Let X be a birack. We denote by $\mathrm{Ret}(X)$ the quotient birack $X/_{\approx}$. We say that a birack has *multipermutation level* k if k is the smallest positive integer such that $|\mathrm{Ret}^k(X)| = 1$.

Observation

A birack X has multipermutation level 1 if and only if there exist permutations $f, g \in S_X$ such that $fg = gf$ and $L_x = f$ and $\mathbf{R}_x = g$, for each $x \in X$.

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Nilpotency vs. multipermutation, idempotent version

Theorem (P.J.,A.P.,A.Z.-D.)

Let X be a distributive **idempotent** *birack. Then TFAE*

- *X has multipermutation level k,*
- *$\mathrm{Mlt}(X)$ is nilpotent of class $k-1$,*
- *k is the smallest number such that*

$$(\cdots(x_0 \circ x_1) \circ x_2) \circ \cdots) \circ x_k = (\cdots(x_1 \circ x_1) \circ \cdots) \circ x_k$$
$$x_1 \bullet (\cdots \bullet (x_{k-1} \bullet (x_k \bullet x_{k+1}) \cdots) = x_1 \bullet (\cdots \bullet (x_{k-1} \bullet x_k) \cdots)$$

Corollary

Let Q be a quandle. Then Q has multipermutation level k if and only if $\mathrm{LMlt}(Q)$ is nilpotent of class $k-1$.

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Corollary

Let Q be a quandle. Then Q has multipermutation level k if and only if $\mathrm{LMlt}(Q)$ *is nilpotent of class* $k-1$.

Nilpotency vs. multipermutation, non-idempotent version

Theorem (P.J.,A.P.,A.Z.-D.)

Let X be a distributive birack and let $k \geqslant 2$. Then TFAE

- *X has multipermutation level k,*
- $\mathrm{Mlt}(X)$ *is nilpotent of class $k-1$,*
- *k is the smallest number such that*

$$(\cdots(x_0 \circ x_1) \circ \cdots) \circ x_k = (\cdots(x_0' \circ x_1) \circ \cdots) \circ x_k$$
$$x_1 \bullet (\cdots \bullet (x_k \bullet x_{k+1}) \cdots) = x_1 \bullet (\cdots \bullet (x_k \bullet x_{k+1}') \cdots)$$