Yang–Baxter equation and a congruence of biracks

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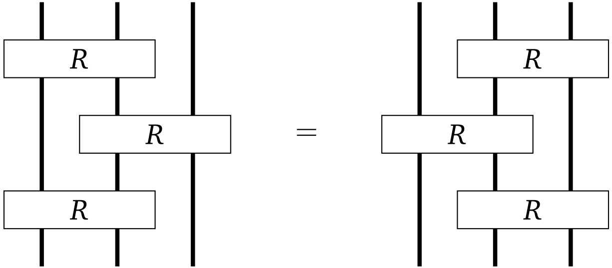
Yang–Baxter equation

Definition

Let V be a vector space. A homomorphism $R : V \otimes V \to V \otimes V$ is called a *solution of Yang–Baxter equation* if it satisfies

$$(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R).$$



Set-theoretic solutions

Definition

Let X be a set. A mapping $r : X \times X \to X \times X$ is called a *set-theoretic solution of Yang–Baxter equation* if it satisfies

$$(r \times \mathrm{id}_X)(\mathrm{id}_X \times r)(r \times \mathrm{id}_X) = (\mathrm{id}_X \times r)(r \times \mathrm{id}_X)(\mathrm{id}_X \times r).$$

A solution $r : (x, y) \mapsto (\sigma_x(y), \tau_y(x))$ is called *non-degenerate* if σ_x and τ_y are bijections, for all $x, y \in X$. A solution is called *involutive* if $r^2 = \mathrm{id}_{X^2}$.

Observation

If r is involutive then $\tau_y(x) = \sigma^{-1}_{\sigma_x(y)}(x)$.

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Examples of solutions

Example

If $\tau_y = \mathrm{id}_X$, for all $y \in X$, then (σ, τ) is a solution if and only if σ_x is a homomorphism, for all $x \in X$, that means

$$\sigma_x(\sigma_y(z)) = \sigma_{\sigma_x(y)}\sigma_x(z).$$

Example

If $\sigma_{\sigma_x(y)} = \sigma_y = \tau_y^{-1}$ then (σ, τ) is an involutive solution.

Example

σ	1	2	3
1	1	2	3
2	1	2	3
3	2	1	3

τ	1	2	3
1	1	1	2
2	2	2	1
3	3	3	3

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Vocabulary

universal algebra setting	**STSYBE setting**
support of a solution	quadratic set
identity	condition
idempotent	square-free
subsolution	restricted solution
(left) ideal	(left) invariant subset
projection algebra	trivial solution

Retraction relation

Definition

Let r be an involutive solution on a set X. We define a relation $\sim$ on X as

$$x \sim y \text{ if and only if } \sigma_x = \sigma_y.$$

Theorem (Etingof, Schedler, Soloviev)

Let r be an involutive solution on a set X. Then there is a well-defined involutive solution on the set $X/\sim$.

Sketch of the proof.

- Define a group $G = \langle X;\ xy = \sigma_x(y)\tau_y(x)\rangle$.
- Prove that $f : x \mapsto \sigma_x$ is a group homomorphism.
- Clearly $x \sim y$ if and only if $f(x) = f(y)$.
- The group $G/\operatorname{Ker} f$ corresponds to the solution $X/\sim$. $\square$

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Definition of a birack

Definition

A *birack* is an algebra $(X, \circ, \bullet, \backslash, /)$ that satisfies

$$x\backslash(x \circ y) = y, \qquad (x \bullet y)/y = x,$$

$$x \circ (x\backslash y) = y, \qquad (x/y) \bullet y = x,$$

$$x \circ (y \circ z) = (x \circ y) \circ ((x \bullet y) \circ z),$$

$$(x \circ y) \bullet ((x \bullet y) \circ z) = (x \bullet (y \circ z)) \circ (y \bullet z),$$

$$(x \bullet y) \bullet z = (x \bullet (y \circ z)) \bullet (y \bullet z),$$

A birack is said to be *involutive* if it satisfies

$$(x \circ y) \circ (x \bullet y) = x, \qquad (x \circ y) \bullet (x \bullet y) = y.$$

Observation

If $(X, \circ, \bullet, \backslash, /)$ *is a birack then* $(x \circ y, x \bullet y)$ *is a solution.*
Conversely, if (σ, τ) *is a solution then, by setting* $x \circ y = \sigma_x(y)$, $x \bullet y = \tau_y(x)$, $x\backslash y = \sigma_x^{-1}(y)$ *and* $x/y = \tau_y^{-1}(x)$, *we obtain a birack.*

Retraction relation of biracks

Definition

Let $(X, \circ, \bullet, \backslash, /)$ be a birack. We define a relation $\sim$ on X as follows:

$$x \sim y \text{ if and only if } x \circ z = y \circ z, \text{ for all } z \in X.$$

Theorem (Etingof, Schedler, Soloviev)

If a birack is involutory then $\sim$ is a congruence.

Proposition (P. J., A. P., A. Z.-D.)

If $\circ$ is left distributive, i.e., $x \circ (y \circ z) = (x \circ y) \circ (x \circ z)$, then $\sim$ is a congruence.

Fact

There exists a birack, for which $\sim$ is not a congruence.

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Retraction congruence of biracks

Definition

Let $(X, \circ, \bullet, \backslash, /)$ be a birack. We define a relation $\approx$ on X as follows:

$$x \approx y \text{ if and only if } x \circ z = y \circ z \text{ and } z \bullet x = z \bullet y, \text{ for all } z \in X.$$

Theorem (P. J., A. P., A. Z.-D.)

$\approx$ is a congruence of every birack.

Proof.

For each $x \approx x'$, $y \approx y'$ and $z \in X$, we prove

$$(x \circ y) \circ z = (x' \circ y') \circ z \qquad z \bullet (x \circ y) = z \bullet (x' \circ y')$$
$$(x \bullet y) \circ z = (x' \bullet y') \circ z \qquad z \bullet (x \bullet y) = z \bullet (x' \bullet y')$$
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