Subquandles of Affine Quandles

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Gdańsk, 15th June 2018

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CZECH UNIVERSITY OF LIFE SCIENCES PRAGUE

https://authors.elsevier.com/a/1XDB44~FOykxI

Definitions

Definition

A binary algebra $(Q,*)$ is called a *quandle* if it satisfies

- $x*x=x,$ (idempotency)
- $(x*y)*z=(x*z)*(y*z),$ (right distributivity)
- $\forall a,b\ \exists!x\colon x*a=b.$ (right quasigroup)

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A quandle $(Q,*)$ is called *affine* (or *Alexander*) if there exists an abelian group operation $+$ on Q and an automorphism f of $(Q,+)$ such that

$$x*y=f(x)-f(y)+y.$$

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Is a quandle affine?

Question:

Given a quandle $(Q, *)$. Does there exist an operation $+$ and a constant $0 \in Q$, such that $(Q, +, 0)$ is an abelian group with the property

$$x * y = f(x) - f(y) + y,$$

for some f, an automorphism of $(Q, +, 0)$?

Q_1	1	2	3	4	5	6
1	1	1	2	2	1	1
2	2	2	1	1	2	2
3	3	3	3	3	4	4
4	4	4	4	4	3	3
5	5	5	6	6	5	5
6	6	6	5	5	6	6

Q_2	1	2	3	4	5	6
1	1	1	1	3	3	3
2	2	2	2	1	1	1
3	3	3	3	2	2	2
4	5	5	5	4	4	4
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Q_5	1	2	3	4	5	6
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2	3	2	1	2	2	2
3	2	1	3	3	3	3
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More definitions

Definition

A quandle is called *quasi-affine* if it embeds into an affine quandle.

Example

- The monoid $(\mathbb{Z}, +, 0)$ is affine.
- The monoid $(\mathbb{N}, +, 0)$ is quasi-affine but not affine.

Definition

A quandle is called *medial* (or *abelian*) if it satisfies

$$(x * y) * (z * u) = (x * z) * (y * u).$$

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Permutation groups

Definitions

Let Q be a quandle. The bijection $R_x : a \mapsto a * x$ is called a *right translation* of Q. The *right multiplication group* (or *inner automorphism group*) of Q is

$$\mathrm{RMlt}(Q) = \langle R_x;\ x \in Q \rangle.$$

The *displacement group* of Q (or *transvection group*) is

$$\mathrm{Dis}(Q) = \langle R_x R_y^{-1};\ x, y \in Q \rangle.$$

Proposition

Both $\mathrm{RMlt}(Q)$ *and* $\mathrm{Dis}(Q)$ *have the same orbits of action on* Q.

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A quandle Q *is medial if and only if* $\mathrm{Dis}(Q)$ *is abelian.*

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$\mathrm{LMlt}(Q_1) \simeq \mathbb{Z}_2^2$

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Characterisation of quasi-affine quandles

Theorem (P.J., A.P., D.S., A.Z.-D.)

Let Q be a quandle. TFAE

- *Q is quasi-affine;*
- *$\mathrm{Dis}(Q)$ is abelian and semiregular;*
- *Q can be constructed as an extension (described on the slide #10).*

Corollary

A finite quandle Q is quasi-affine is and only if $\mathrm{Dis}(Q)$ is abelian and the size of $\mathrm{Dis}(Q)$ is equal to the size of any its orbits.

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Semiregular extension

Definition

Let A be an abelian group, let f be an automorphism of A and let I be a multiset of elements from A. We define an operation $*$ on $I \times A$ as

$$(i,a) * (j,b) = (i, f(a) - f(b) + b + j - i)$$

and we denote the quandle by $\mathrm{Ext}(A, f, I)$.

Lemma

There are, up to isomorphism, 4 quasi-affine quandles of size 6:

- $\mathrm{Ext}(\mathbb{Z}_1, \mathrm{id}_{\mathbb{Z}_1}, \{\{0,0,0,0,0,0\}\})$,
- $\mathrm{Ext}(\mathbb{Z}_2, \mathrm{id}_{\mathbb{Z}_2}, \{\{0,0,1\}\})$,
- $\mathrm{Ext}(\mathbb{Z}_3, \mathrm{id}_{\mathbb{Z}_3}, \{\{0,1\}\})$,
- $\mathrm{Ext}(\mathbb{Z}_3, 2 \cdot \mathrm{id}_{\mathbb{Z}_3}, \{\{0,0\}\})$.

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Let A be an abelian group, let f be an automorphism of A and let I be a multiset of elements from A. We define an operation $*$ on $I \times A$ as

$$(i,a) * (j,b) = (i, f(a) - f(b) + b + j - i)$$

and we denote the quandle by $\mathrm{Ext}(A,f,I)$.

Lemma

There are, up to isomorphism, 4 quasi-affine quandles of size 6:

- $\mathrm{Ext}(\mathbb{Z}_1, \mathrm{id}_{\mathbb{Z}_1}, \{\{0,0,0,0,0,0\}\})$,
- $\mathrm{Ext}(\mathbb{Z}_2, \mathrm{id}_{\mathbb{Z}_2}, \{\{0,0,1\}\})$,
- $\mathrm{Ext}(\mathbb{Z}_3, \mathrm{id}_{\mathbb{Z}_3}, \{\{0,1\}\})$,
- $\mathrm{Ext}(\mathbb{Z}_3, 2 \cdot \mathrm{id}_{\mathbb{Z}_3}, \{\{0,0\}\})$.

Characterisation of finite affine quandles

Theorem (P.J., A.P., D.S., A.Z.-D.)

Let Q be a finite quasi-affine quandle. Then TFAE

- *Q is affine;*
- *every row is balanced,*
  that means, for every $e \in Q$ and every $d, d' \in Qe$, we have $m_{e,d} = m_{e,d'}$;
- *at least one row is balanced,*
  that means, there exists $e \in Q$ such that, for every $d, d' \in Qe$, we have $m_{e,d} = m_{e,d'}$,

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Q_1	1	2	3	4	5	6
1	1	1	2	2	1	1
2	2	2	1	1	2	2
3	3	3	3	3	4	4
4	4	4	4	4	3	3
5	5	5	6	6	5	5
6	6	6	5	5	6	6

$\mathrm{LMlt}(Q_1) \cong \mathbb{Z}_2^2$

$\mathrm{Dis}(Q_1) \cong \mathbb{Z}_2^2$

Q_2	1	2	3	4	5	6
1	1	1	1	3	3	3
2	2	2	2	1	1	1
3	3	3	3	2	2	2
4	5	5	5	4	4	4
5	6	6	6	5	5	5
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$\mathrm{LMlt}(Q_2) \cong \mathbb{Z}_3^2$

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Q_3	1	2	3	4	5	6
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3	2	1	3	2	1	3
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Q_5	1	2	3	4	5	6
1	1	3	2	1	1	1
2	3	2	1	2	2	2
3	2	1	3	3	3	3
4	4	4	4	4	4	4
5	5	5	5	5	5	5
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Constructing the abelian group

Lemma (X. Hou ('12); J. Šťovíček ('13))

Let R be a hereditary ring, let A be an R-module and φ its endomorphism. Then there exist an R-module $E \geqslant A$ and an epimorphism $\psi : E \to A$ such that $\psi|_A = \varphi$ and $\psi/A : E/A \cong A/\operatorname{Im}(\varphi)$.

In our context $R = \mathbb{Z}$ and $\varphi = \mathrm{id}_A - f$.

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Let Q be an affine quandle. How to construct the operation $+$ on Q and the automorphism f?

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Summary of computation

Proposition

Testing (quasi-)affinity runs in $\mathcal{O}(n^3 \cdot \log n)$ with respect to $n = |Q|$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
quasi-af.	1	1	2	3	4	4	6	9	12	7	10	17	12	10	14
affine	1	1	2	3	4	2	6	7	11	4	10	6	12	6	8

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