Subdirectly irreducible medial quandles

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Definition of a medial quandle

Definition

A *medial quandle* is an algebra $(A, *, \backslash)$ satisfying

- $x * x = x$,
- $(x * y) * (z * u) = (x * z) * (y * u)$,
- $x \backslash (x * y) = y$,
- $x * (x \backslash y) = y$.

Observation

Every medial quandle satisfies

$$x \backslash x = x$$

$$(x \backslash y) \backslash (z \backslash u) = (x \backslash z) \backslash (y \backslash u)$$

$$(x \backslash y) * (z \backslash u) = (x * z) \backslash (y * u)$$

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The smallest subdirectly irreducible medial quandles

$*$	1	2
1	1	2
2	1	2

$*$	3	4	5
3	3	5	4
4	5	4	3
5	4	3	5

$*$	6	7	8
6	6	7	8
7	6	7	8
8	7	6	8

Affine quandles

Definition

Let A be an abelian group and let $f \in \mathrm{Aut}(A)$. Define

$$x * y = x + f(y - x) = (1 - f)(x) + f(y)$$
$$x \backslash y = x + f^{-1}(y - x) = (1 - f^{-1})(x) + f^{-1}(y)$$

Then $(A, *, \backslash)$ is a medial quandle called *affine* quandle or *Alexander* quandle and denoted by $\mathrm{Aff}(A, f)$.

Observation

An affine quandle is a reduct of a $\mathbb{Z}[x, x^{-1}]$-module. It is polynomially equivalent to a module if and only if $(1 - f)$ is an automorphism (iff the quandle is a quasigroup).

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Simple quandles

Theorem (D. Joyce (1982))

All simple quandles are

- $\mathrm{Aff}(\mathbb{Z}_2, 1)$,
- $\mathrm{Aff}(A, x)$, *where* A *is a simple* $\mathbb{Z}[x, x^{-1}]$*-module and* $0 \neq xa \neq a$, *for all* $0 \neq a \in A$.

Types of subdirectly irreducible modes

Theorem (K. Kearnes (1999))

Let M be a subdirectly irreducible medial quandle with the monolith μ. Then the type of the interval $[1, \mu]$ is one of the following:

- *type 1 (set type),*
- *type 2 (quasi-affine type).*

Theorem (K. Kearnes (1999))

All subdirectly irreducible medial quandles of type 2 are quasigroups.

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Minimal left ideals

Proposition (P.J., A.P., A.Z.-D.)

Let Q be a medial quandle and let I be one of its minimal left ideals. Then, using operations $*$ and $\backslash$ on Q, we can endow I with a structure of a $\mathbb{Z}[x, x^{-1}]$-module.

Proposition (P.J., A.P., A.Z.-D.)

Let Q be a subdirectly irreducible medial quandle. Then some minimal left ideal is a subdirectly irreducible $\mathbb{Z}[x, x^{-1}]$-module.

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Let Q be a subdirectly irreducible medial quandle. Then some minimal left ideal is a subdirectly irreducible $\mathbb{Z}[x,x^{-1}]$-module.

Divisible SIMQ

Lemma

Every divisible medial quandle is affine.

Example

Let p be a prime and

$$\mathbb{Z}_{p^\infty} = \left\{ \left[\frac{a}{p^k} \right]_\sim ; \, a, k \in \mathbb{N} \right\},$$

where $\frac{a}{p^k} \sim \frac{b}{p^n}$ iff $ap^n \equiv bp^k \pmod{p^{k+n}}$.
Then $(\mathbb{Z}_{p^\infty}, 1-p)$ is a subdirectly irreducible affine quandle of the set type.

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Construction of SIMQ

Theorem (P.J., A.P., A.Z.-D.)

Let A be a subdirectly irreducible $\mathbb{Z}[x,x^{-1}]$-module.
Suppose that the endomorphism $\varphi : a \mapsto a - xa$ is not injective. Let C be a (non-empty) subset of a transversal to $\varphi(A)$ in A such that $\varphi(A) \cup C$ generates A.
We define an operation $*$ on $Q = A \cup (\varphi(A) \times C)$ as follows:

$$
\begin{aligned}
a * b &= a - xa + xb \\
(a,c) * (b,d) &= (xb + (1-x)\cdot(a+c-d), d) \\
(a,c) * b &= a + xb + c \\
a * (b,d) &= (xb + (1-x)\cdot(a - xa - d), d).
\end{aligned}
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Then $(Q,*)$ is a subdirectly irreducible medial quandle of the set type. On the other hand, every (except of one) non-divisible subdirectly irreducible medial quandle can be obtained this way.

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Example of a SIMQ

$A = \mathbb{Z}_4$, $x = 3$, $\varphi = 2$, $C = \{0, 1\}$.

$*$	0	1	2	3	(0, 0)	(2, 0)	(0, 1)	(2, 1)
0	0	3	2	1	(0, 0)	(2, 0)	(2, 1)	(0, 1)
1	2	1	0	3	(0, 0)	(2, 0)	(2, 1)	(0, 1)
2	0	3	2	1	(0, 0)	(2, 0)	(2, 1)	(0, 1)
3	2	1	0	3	(0, 0)	(2, 0)	(2, 1)	(0, 1)
(0, 0)	0	3	2	1	(0, 0)	(2, 0)	(2, 1)	(0, 1)
(2, 0)	2	1	0	3	(0, 0)	(2, 0)	(2, 1)	(0, 1)
(0, 1)	1	0	3	2	(2, 0)	(0, 0)	(0, 1)	(2, 1)
(2, 1)	3	2	1	0	(2, 0)	(0, 0)	(0, 1)	(2, 1)

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Involutory medial quandles

Definition

A groupoid is called involutory if $x * (x * y) = y$.

Lemma

An affine quandle is involutory iff it is $\mathrm{Aff}(A, -1)$.

Proposition (P.J., A.P., A.Z.-D.)

All SI involutory medial quandles are obtained via

- $\mathrm{Aff}(\mathbb{Z}_{p^k}, -1)$, *where* p *is an odd prime,* $k \in \{1, 2, 3, \ldots, \infty\}$;
- *construction with* $A = \mathbb{Z}_{2^k}$, $k \in \mathbb{N}^+$, $x = -1$, $C = \{1\}$;
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