Subdirectly irreducible medial quandles

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New Orchard, 18th June 2017





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Definition of a medial quandle

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A medial quandle is an algebra $(A, *, \setminus)$ satisfying

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$$x * x = x$$
,

•
$$(x * y) * (z * u) = (x * z) * (y * u),$$

•
$$x \setminus (x * y) = y$$
,

•
$$x * (x \setminus y) = y$$

Observation

Every medial quandle satisfies

$$\begin{aligned} x \setminus x &= x \\ (x \setminus y) \setminus (z \setminus u) &= (x \setminus z) \setminus (y \setminus u) \\ (x \setminus y) &* (z \setminus u) &= (x * z) \setminus (y * u) \end{aligned}$$

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Subdirectly irreducible medial quandles

Basic definitions

The smallest subdirectly irreducible medial quandles

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Affine quandles

Definition

Let *A* be an abelian group and let $f \in Aut(A)$. Define

$$\begin{aligned} x * y &= x + f(y - x) = (1 - f)(x) + f(y) \\ x \setminus y &= x + f^{-1}(y - x) = (1 - f^{-1})(x) + f^{-1}(y) \end{aligned}$$

Then $(A, *, \setminus)$ is a medial quandle called *affine* quandle or *Alexander* quandle and denoted by Aff(A, f).

Observation

An affine quandle is a reduct of a $\mathbb{Z}[x, x^{-1}]$ -module. It is polynomially equivalent to a module if and only if (1 - f) is an automorphism (iff the quandle is a quasigroup).

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Subdirectly irreducible medial quandles Subdirectly irreducible medial quasigroups

Simple quandles

Theorem (D. Joyce (1982))

All simple quandles are

- Aff($\mathbb{Z}_2, 1$),
- Aff(A, x), where A is a simple $\mathbb{Z}[x, x^{-1}]$ -module and $0 \neq xa \neq a$, for all $0 \neq a \in A$.

Types of subdirectly irreducible modes

Theorem (K. Kearnes (1999))

Let *M* be a subdirectly irreducible medial quandle with the monolith μ . Then the type of the interval $[1, \mu]$ is one of the following:

- type 1 (set type),
- type 2 (quasi-affine type).

Theorem (K. Kearnes (1999))

All subdirectly irreducible medial quandles of type 2 are quasigroups.

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Minimal left ideals

Proposition (P.J., A.P., A.Z.-D.)

Let Q be a medial quandle and let I be one of its minimal left ideals. Then, using operations * and \setminus on Q, we can endow I with a structure of a $\mathbb{Z}[x, x^{-1}]$ -module.

Proposition (P.J., A.P., A.Z.-D.)

Let Q be a subdirectly irreducible medial quandle. Then some minimal left ideal is a subdirectly irreducible $\mathbb{Z}[x, x^{-1}]$ -module.

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Divisible SIMQ

Lemma

Every divisible medial quandle is affine.

Example

Let *p* be a prime and

$$\mathbb{Z}_{p^{\infty}} = \left\{ \left[rac{a}{p^k}
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where $\frac{a}{p^k} \sim \frac{b}{p^n}$ iff $ap^n \equiv bp^k \pmod{p^{k+n}}$. Then $(\mathbb{Z}_{p^{\infty}}, 1-p)$ is a subdirectly irreducible affine quandle of the set type.

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Theorem (P.J., A.P., A.Z.-D.)

Let A be a subdirectly irreducible $\mathbb{Z}[x, x^{-1}]$ -module.

Suppose that the endomorphism $\varphi : a \mapsto a - xa$ is not injective. Let C be a (non-empty) subset of a transversal to $\varphi(A)$ in A such that $\varphi(A) \cup C$ generates A.

We define an operation * on $Q = A \cup (\varphi(A) \times C)$ as follows:

$$a * b = a - xa + xb$$

(a, c) * (b, d) = (xb + (1 - x) \cdot (a + c - d), d)
(a, c) * b = a + xb + c
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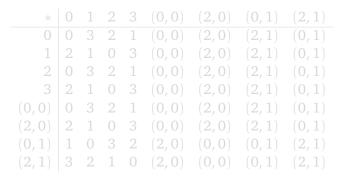
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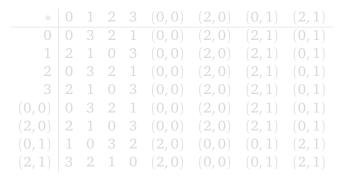
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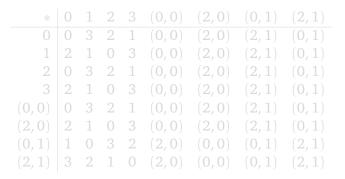
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					(0,0)			
1	2	1	0	3	(0,0)	(2, 0)	(2, 1)	(0, 1)
2	0	3	2	1	(0,0)	(2, 0)	(2, 1)	(0, 1)
3	2	1	0	3	(0,0)	(2, 0)	(2, 1)	(0, 1)
(0,0)	0	3	2	1	(0,0)	(2, 0)	(2, 1)	(0, 1)
(2,0)	2	1	0	3	(0,0)	(2, 0)	(2, 1)	(0, 1)
(0, 1)	1	0	3	2	(2, 0)	(0, 0)	(0, 1)	(2, 1)
(2, 1)	3	2	1	0	(2, 0)	(0, 0)	(0, 1)	(2, 1)

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Involutory medial quandles

Definition

A groupoid is called involutory if
$$x * (x * y) = y$$
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Lemma

An affine quandle is involutory iff it is Aff(A, -1).

Proposition (P.J., A.P., A.Z.-D.)

All SI involutory medial quandles are obtained via

- Aff(\mathbb{Z}_{p^k} , -1), where p is an odd prime, $k \in \{1, 2, 3, \dots, \infty\}$;
- construction with $A = \mathbb{Z}_{2^k}$, $k \in \mathbb{N}^+$, x = -1, $C = \{1\}$;
- construction with $A = \mathbb{Z}_{2^k}$, $k \in \mathbb{N}^+$, x = -1, $C = \{0, 1\}$;
- Aff($\mathbb{Z}_2, -1$) and Aff($\mathbb{Z}_{2^{\infty}}, -1$);
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