Subquandles of Affine Quandles

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Definitions

Definition

A binary algebra $(Q,*)$ is called a *quandle* if it satisfies

- $x*x=x$, (idempotency)
- $x*(y*z)=(x*y)*(x*z)$, (left distributivity)
- $\forall a,b\ \exists! x\colon a*x=b$. (left quasigroup)

Definition

A quandle $(Q,*)$ is called *affine* (or *Alexander*) if there exists an abelian group operation $+$ on Q and an automorphism f of $(Q,+)$ such that

$$x*y=x-f(x)+f(y).$$

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A binary algebra $(Q, *)$ is called a *quandle* if it satisfies

- $x * x = x$, (idempotency)
- $x * (y * z) = (x * y) * (x * z)$, (left distributivity)
- $\forall a, b\ \exists! x$: $a * x = b$. (left quasigroup)

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$$x * y = x - f(x) + f(y).$$

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3	2	1	3	4	6	5
4	2	1	3	4	6	5
5	1	2	4	3	5	6
6	1	2	4	3	5	6

Q_2	1	2	3	4	5	6
1	1	2	3	5	6	4
2	1	2	3	5	6	4
3	1	2	3	5	6	4
4	3	1	2	4	5	6
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More definitions

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A quandle is called *quasi-affine* if it embeds into an affine quandle.

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A quandle is called *medial* (or *abelian*) if it satisfies

$$(x * y) * (z * u) = (x * z) * (y * u).$$

Definition

An algebra A is called *abelian* (in this talk *UA-abelian*) if the set $\{(a, a);\ a \in A\}$ is a class of a congruence of $A \times A$.

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Permutation groups

Definitions

Let Q be a quandle. The bijection $L_x : a \mapsto x * a$ is called a *left translation* of Q. The *left multiplication group* (or *inner automorphism group*) of Q is

$$\mathrm{LMlt}(Q) = \langle L_x;\ x \in Q \rangle.$$

The *displacement group* of Q (or *transvection group*) is

$$\mathrm{Dis}(Q) = \langle L_x L_y^{-1};\ x, y \in Q \rangle.$$

Proposition

Both $\mathrm{LMlt}(Q)$ *and* $\mathrm{Dis}(Q)$ *have the same orbits of action on* Q.

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A quandle Q *is medial if and only if* $\mathrm{Dis}(Q)$ *is abelian.*

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6	1	2	4	3	5	6

$\mathrm{LMlt}(Q_1) \cong \mathbb{Z}_2^2$

$\mathrm{Dis}(Q_1) \cong \mathbb{Z}_2^2$

Q_2	1	2	3	4	5	6
1	1	2	3	5	6	4
2	1	2	3	5	6	4
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4	3	1	2	4	5	6
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Characterisation of quasi-affine quandles

Theorem (P.J., A.P., D.S., A.Z.-D.)

Let Q be a quandle. TFAE

- *Q is quasi-affine;*
- $\mathrm{Dis}(Q)$ *is abelian and semiregular;*
- *Q can be constructed as an extension (described on the slide #9);*
- *Q is UA-abelian.*

Corollary

A finite quandle Q is quasi-affine is and only if $\mathrm{Dis}(Q)$ *is abelian and the size of* $\mathrm{Dis}(Q)$ *is equal to the size of any its orbits.*

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Semiregular extension

Definition

Let A be an abelian group, let f be an automorphism of A and let I be a multiset of elements from A. We define an operation $*$ on $I \times A$ as

$$(i,a) * (j,b) = (j, a - f(a) + f(b) + i - j)$$

and we denote the quandle by $\mathrm{Ext}(A,f,I)$.

Example

$\mathrm{Aff}(A,f) \cong \mathrm{Ext}(\mathrm{Im}(\mathrm{id}_A - f), f|_{\mathrm{Im}(\mathrm{id}_A - f)}, (\mathrm{id}_A - f)(T))$, where T is a transversal to $\mathrm{Im}(\mathrm{id}_A - f)$ in A.

Lemma

There are, up to isomorphism, 4 quasi-affine quandles of size 6:

- $\mathrm{Ext}(\mathbb{Z}_1, \mathrm{id}_{\mathbb{Z}_1}, \{\{0,0,0,0,0,0\}\})$,
- $\mathrm{Ext}(\mathbb{Z}_2, \mathrm{id}_{\mathbb{Z}_2}, \{\{0,0,1\}\})$,
- $\mathrm{Ext}(\mathbb{Z}_3, \mathrm{id}_{\mathbb{Z}_3}, \{\{0,1\}\})$,
- $\mathrm{Ext}(\mathbb{Z}_3, 2 \cdot \mathrm{id}_{\mathbb{Z}_3}, \{\{0,0\}\})$.

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There are, up to isomorphism, 4 quasi-affine quandles of size 6:

- $\mathrm{Ext}(\mathbb{Z}_1, \mathrm{id}_{\mathbb{Z}_1}, \{\{0, 0, 0, 0, 0, 0\}\})$,
- $\mathrm{Ext}(\mathbb{Z}_2, \mathrm{id}_{\mathbb{Z}_2}, \{\{0, 0, 1\}\})$,
- $\mathrm{Ext}(\mathbb{Z}_3, \mathrm{id}_{\mathbb{Z}_3}, \{\{0, 1\}\})$,
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Semiregular extension

Definition

Let A be an abelian group, let f be an automorphism of A and let I be a multiset of elements from A. We define an operation $*$ on $I \times A$ as

$$(i,a) * (j,b) = (j, a - f(a) + f(b) + i - j)$$

and we denote the quandle by $\mathrm{Ext}(A,f,I)$.

Example

$\mathrm{Aff}(A,f) \cong \mathrm{Ext}(\mathrm{Im}(\mathrm{id}_A - f), f|_{\mathrm{Im}(\mathrm{id}_A - f)}, (\mathrm{id}_A - f)(T))$, where T is a transversal to $\mathrm{Im}(\mathrm{id}_A - f)$ in A.

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Characterisation of affine quandles

Definition

Let G be a group and $H \leqslant G$. We say that a multiset T is a *left multi-transversal* of H in G, if $|T \cap aH| = |T \cap bH|$, for each $a, b \in G$. The cardinality of $|T \cap H|$ is called the *muliplicity* of T.

Theorem (P.J., A.P., D.S., A.Z.-D.)

Let Q be a quasi-affine quandle. Then TFAE:

- *Q is affine;*
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- *whenever Q is isomorphic to some $\mathrm{Ext}(A, f, T)$, where A is an abelian group and $f \in \mathrm{Aut}(A)$ then T is a multi-transversal of $\mathrm{Im}(\mathrm{id}_A - f)$ in $\langle \mathrm{Im}(\mathrm{id}_A - f) \cup T \rangle$.*

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Examples of extensions

- $\mathrm{Ext}(\mathbb{Z}_1, \mathrm{id}_{\mathbb{Z}_1}, \{\{0,0,0,0,0,0\}\})$:
  $\{\{0,0,0,0,0,0\}\}$ is a multi-transversal of $\{0\}$ in $\{0\}$ of multiplicity 6.
  This quandle is affine.
- $\mathrm{Ext}(\mathbb{Z}_2, \mathrm{id}_{\mathbb{Z}_2}, \{\{0,0,1\}\})$:
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  This quandle is affine.

Characterisation of finite affine quandles

Theorem (P.J., A.P., D.S., A.Z.-D.)

Let Q be a finite quasi-affine quandle. Then TFAE

- *Q is affine;*
- *for every $e \in Q$ and every $d, d' \in Qe$, we have $m_{e,d} = m_{e,d'}$;*
- *there exists $e \in Q$ such that, for every $d, d' \in Qe$, we have $m_{e,d} = m_{e,d'}$,*

where $m_{x,y} = |\{z \in Q;\ xz = y\}|$.

Q_1	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	2	3	4	5	6
3	2	1	3	4	6	5
4	2	1	3	4	6	5
5	1	2	4	3	5	6
6	1	2	4	3	5	6

$\mathrm{LMlt}(Q_1) \cong \mathbb{Z}_2^2$

$\mathrm{Dis}(Q_1) \cong \mathbb{Z}_2^2$

Q_2	1	2	3	4	5	6
1	1	2	3	5	6	4
2	1	2	3	5	6	4
3	1	2	3	5	6	4
4	3	1	2	4	5	6
5	3	1	2	4	5	6
6	3	1	2	4	5	6

$\mathrm{LMlt}(Q_2) \cong \mathbb{Z}_3^2$

$\mathrm{Dis}(Q_2) \cong \mathbb{Z}_3$

Q_3	1	2	3	4	5	6
1	1	3	2	4	6	5
2	3	2	1	6	5	4
3	2	1	3	5	4	6
4	1	3	2	4	6	5
5	3	2	1	6	5	4
6	2	1	3	5	4	6

$\mathrm{LMlt}(Q_3) \cong S_3$

$\mathrm{Dis}(Q_3) \cong A_3$

Q_4	1	2	3	4	5	6
1	1	2	4	3	6	5
2	1	2	4	3	6	5
3	2	1	3	4	5	6
4	2	1	3	4	5	6
5	2	1	3	4	5	6
6	2	1	3	4	5	6

$\mathrm{LMlt}(Q_4) \cong \mathbb{Z}_2^2$

$\mathrm{Dis}(Q_4) \cong \mathbb{Z}_2$

Q_5	1	2	3	4	5	6
1	1	3	2	4	5	6
2	3	2	1	4	5	6
3	2	1	3	4	5	6
4	1	2	3	4	5	6
5	1	2	3	4	5	6
6	1	2	3	4	5	6

$\mathrm{LMlt}(Q_5) = S_3$

$\mathrm{Dis}(Q_5) = S_3$

Q_6	1	2	3	4	5	6
1	1	2	4	3	6	5
2	1	2	4	3	6	5
3	2	1	3	4	6	5
4	2	1	3	4	6	5
5	3	4	2	1	5	6
6	3	4	2	1	5	6

$\mathrm{LMlt}(Q_6) \cong D_8$

$\mathrm{Dis}(Q_6) \cong \mathbb{Z}_4$

Q_1	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	2	3	4	5	6
3	2	1	3	4	6	5
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5	1	2	4	3	5	6
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Constructing the abelian group

Lemma (X. Hou ('12); J. Šťovíček ('13))

Let R be a hereditary ring, let A be an R-module and φ its endomorphism. Then there exist an R-module $E \geqslant A$ and an epimorphism $\psi : E \to A$ such that $\psi|_A = \varphi$ and $\psi/A : E/A \cong A/\operatorname{Im}(\varphi)$.

In our context $R = \mathbb{Z}$ and $\varphi = \mathrm{id}_A - f$.

Open Problem

Let Q be an affine quandle. How to construct the operation $+$ on Q and the automorphism f?

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Let Q be an affine quandle. How to construct the operation $+$ on Q and the automorphism f?

Summary of computation

Proposition

Testing (quasi-)affinity runs in $\mathcal{O}(n^3 \cdot \log n)$ with respect to $n = |Q|$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
quasi-af.	1	1	2	3	4	4	6	9	12	7	10	17	12	10	14
affine	1	1	2	3	4	2	6	7	11	4	10	6	12	6	8

Thank you for your attention

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Isomorphisms of affine quandles

Proposition

Let $Q = \mathrm{Ext}(A, f, T)$ and $Q' = \mathrm{Ext}(A', f', T')$ be two affine quandles. Then Q and Q' are isomorphic if and only if

- *there exists an isomorphism $\psi : A \mapsto A'$ such that $\psi f = f' \psi$,*
- *T and T' have the same multiplicities.*

Corollary (X. Hou ('11))

Let A and A' be abelian groups, let $f \in \mathrm{Aut}(A)$ and $f' \in \mathrm{Aut}(A')$. The affine quandles $\mathrm{Aff}(A, f)$ and $\mathrm{Aff}(A', f')$ are isomorphic if and only if

- $|A| = |A'|$,
- $\exists \psi : \mathrm{Im}(\mathrm{id}_A - f) \cong \mathrm{Im}(\mathrm{id}_{A'} - f')$ *such that* $\psi f = f' \psi$,
- $|\mathrm{Ker}(\mathrm{id}_A - f) / \mathrm{Ker}(\mathrm{id}_A - f) \cap \mathrm{Im}(\mathrm{id}_A - f)| = |\mathrm{Ker}(\mathrm{id}_{A'} - f') / \mathrm{Ker}(\mathrm{id}_{A'} - f') \cap \mathrm{Im}(\mathrm{id}_{A'} - f')|$.

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- $\exists \psi : \mathrm{Im}(\mathrm{id}_A - f) \cong \mathrm{Im}(\mathrm{id}_{A'} - f')$ *such that* $\psi f = f'\psi$*,*
- $|\mathrm{Ker}(\mathrm{id}_A - f)/\mathrm{Ker}(\mathrm{id}_A - f) \cap \mathrm{Im}(\mathrm{id}_A - f)| = |\mathrm{Ker}(\mathrm{id}_{A'} - f')/\mathrm{Ker}(\mathrm{id}_{A'} - f') \cap \mathrm{Im}(\mathrm{id}_{A'} - f')|$.