Quandles médiaux

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Generalizations of abelian groups

Definitions

Let $\mathcal{A} = (A, \sigma)$ be an algebra. We say that $\mathcal{A}$ is

- *affine* if there exist a ring R, a binary operation $+$ on A, an operation $\cdot : R \times A \to A$ and constants $0, 1 \in A$ such that $(A, +, \cdot, 0, 1)$ is a R-module and all operation from σ can be derived from the module operations;
- *quasi-affine* if $\mathcal{A}$ embeds into an affine algebra;
- *abelian* if there exist an algebra $\mathcal{B}$, a homomorphism $h : \mathcal{A}^2 \to \mathcal{B}$ and an element $c \in \mathcal{B}$ such that $\{(a, a);\ a \in \mathcal{A}\} = h^{-1}(c)$;
- *entropic* if, for each $f, g \in \sigma$ and $x_{i,j} \in \mathcal{A}$ we have

$$f(g(x_{1,1}, \dots, x_{1,n}), \dots, g(x_{k,1}, \dots, x_{k,n})) = \\ = g(f(x_{1,1}, \dots, x_{k,1}), \dots, f(x_{1,n}, \dots, x_{k,n})).$$

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Definition of quandles

Definition

A groupoid $(Q,*)$ is called a *quandle*, if it satisfies

- $x * x = x$, *(idempotency)*
- $(x*y)*z = (x*z)*(y*z)$, *(right distributivity)*
- $\forall y,z\ \exists! x;\quad x*y = z$. *(right quasigroup)*

Fact

The right quasigroup property can be alternatively expressed as follows:
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Examples of quandles

Example (Left zero band)

The groupoid $(Q, *)$ with the operation $x * y = x$.

Example (Group conjugation)

Let $(G, \cdot)$ be a group and let $a * b = b^{-1} \cdot a \cdot b$.

Theorem (D. Joyce)

The knot quandle is a classifying invariant of knots.

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Left translations

Definition

Let $(Q, *)$ be a groupoid. The mapping $R_x : a \mapsto a * x$ is called the *right translation* by x.

Definition

A groupoid Q is called a quandle if it satisfies

- R_x is an endomorphism, for each $x \in Q$, (right distributivity)
- R_x is a permutation, for each $x \in Q$, (right quasigroup)
- x is a fixed point of R_x, for each $x \in Q$. (idempotency)

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Permutation groups

Definitions

- The *right multiplication group* of Q is the permutation group $\mathrm{RMlt}(Q) = \langle R_x;\ x \in Q\rangle$.
- The *displacement group* of Q is the permutation group $\mathrm{Dis}(Q) = \langle R_x R_y^{-1};\ x, y \in Q\rangle$.

Proposition

- $\mathrm{RMlt}(Q)' \trianglelefteq \mathrm{Dis}(Q) \trianglelefteq \mathrm{RMlt}(Q)$,
- *the group* $\mathrm{RMlt}(Q)/\,\mathrm{Dis}(Q)$ *is cyclic,*
- *the natural actions of* $\mathrm{RMlt}(Q)$ *and* $\mathrm{Dis}(Q)$ *on* Q *have the same orbits.*

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Medial quandles

Definition

A groupoid is called *medial*, if it satisfies

$$(x * y) * (u * z) = (x * u) * (y * z)$$

Fact

A quandle is entropic if and only if it is medial. Such quandles are sometimes called abelian.

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A quandle is medial if and only if $\mathrm{Dis}(Q)$ *is abelian.*

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Affine quandles

Fact

A quandle $(Q,*)$ is affine is and only if it is a reduct of a $\mathbb{Z}[x,x^{-1}]$-module, i.e., there exists an abelian group A and an automorphism $f \in \mathrm{Aut}(A)$ such that

$$x*y = f(x-y)+y = f(x)+(1-f)(y).$$

Such a quandle is often called an Alexander *quandle.*

Fact

Let $Q = \mathrm{Aff}(A,f)$ then $\mathrm{Dis}\, Q = \{x \mapsto x+a;\ \forall a \in \mathrm{Im}(1-f)\}$.

Corollary

An affine quandle is medial.

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Abelian quandles

Theorem (P.J.,A.P.,D.S.,A.Z.)

Let Q be a quandle. Then the following conditions are equivalent:

- *Q is abelian,*
- $\mathrm{Dis}(Q)$ *is abelian and semi-regular,*
- *there exist an abelian group A, an automorphism $f \in \mathrm{Aut}(A)$, an index set $\mathcal{I}$ and constants d_i, $i \in \mathcal{I}$, such that*
  - $A = \langle \mathrm{Im}(1-f) \cup \{d_i - d_j;\ \text{for } i,j \in \mathcal{I}\}\rangle$,
  - $Q \cong (A \times \mathcal{I}, *)$ *with the operation $*$ defined as-*

$$(a,i) * (b,j) = (f(a) + (1-f)(b) + d_i - d_j, j).$$

This construction is denoted by $\mathrm{Ab}(A, f, (d_i)_{i \in \mathcal{I}})$.

Corollaries of abelian characterisation

Corollary

A finite quandle is abelian if and only if $\mathrm{Dis}(Q)$ *is abelian and* $|\mathrm{Dis}(Q)| = |Qe|$, *for each* $e \in Q$.

Corollary

Affine and quasi-affine quandles are abelian.

Example (P.J.,A.P.,A.Z.)

The free n-generated medial quandle is $\mathrm{Ab}(\mathbb{Z}[x,x^{-1}]^{n-1}, x, (d_i)_{0\leqslant i<n})$, where $d_0 = 0$ and $\{d_i;\ 1 \leqslant i < n\}$ is a free basis of $\mathbb{Z}[x,x^{-1}]$.

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Isomorphism of affine quandles

Proposition (P.J.,A.P.,D.S.,A.Z.)

Let A_1, A_2 be two abelian groups. Let, for $k \in \{1,2\}$, be $f_k \in \mathrm{Aut}\, A_k$, let $\mathcal{I}_k$ be index sets and $d_{i,k} \in A_k$, for all $i \in \mathcal{I}_k$. Then $\mathrm{Ab}(A_1, f_1, (d_{i,1})_{i\in\mathcal{I}_1})$ is isomorphic to $\mathrm{Ab}(A_2, f_2, (d_{i,2})_{i\in\mathcal{I}_2})$ if and only if

- *there exists ψ, an isomorphism $A_1 \to A_2$;*
- *there exists a bijection $\pi : \mathcal{I}_1 \to \mathcal{I}_2$ such that $\psi \circ f_1 = f_2 \circ \psi$;*
- *there exist a constant $a \in A_2$ and constants $b_j \in \mathrm{Im}(1 - f_2)$, for $j \in \mathcal{I}_2$, such that $\psi(d_{i,1}) = d_{\pi(i),2} + a + b_{\pi(i)}$, for all $i \in \mathcal{I}_1$.*

Affine criterion

Definition

Let G be a group and H its subgroup. A multiset T is called a *left multi-transversal* of H in G if $|xH \cap T| = |H \cap T|$, for each $x \in G$.

Theorem (P.J.,A.P.,D.S.,A.Z.)

Let Q be an abelian quandle. Then TFCAE:

- *Q is affine,*
- *$Q \cong \mathrm{Ab}(A, f, (d_i)_{i\in\mathfrak{I}})$ and $\{d_i;\ i \in \mathfrak{I}\}$ is a multi-transversal of $\mathrm{Im}(1-f)$ in A,*
- *whenever $Q \cong \mathrm{Ab}(A, f, (d_i)_{i\in\mathfrak{I}})$ then $\{d_i;\ i \in \mathfrak{I}\}$ is a multi-transversal of $\mathrm{Im}(1-f)$ in A.*

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Consequences of the affine criterion

Proposition (P.J.,A.P.,D.S.,A.Z.)

A finite quandle Q is affine if and only if

- $\mathrm{Dis}(Q)$ *is abelian and semi-regular,*
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An *indecomposable affine mesh* is an n-tuplet of abelian groups $A_1, \dots, A_n$, together with homomorphisms $\varphi_{i,j} : A_i \to A_j$ and constants $c_{i,j} \in A_j$, for $i, j \in [1, \cdots, n]$, satisfying

(M1) $(1 - \varphi_{i,i}) \in \mathrm{Aut}(A_i)$;

(M2) $c_{i,i} = 0$;

(M3) $\varphi_{j,k} \circ \varphi_{i,j} = \varphi_{j',k} \circ \varphi_{i,j'}$;

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Sums of affine meshes

Definition

The *sum* of an indecomposable affine mesh $\mathcal{A} = (A_i, \varphi_{i,j}, c_{i,j})$ over a set I is the groupoid $\left(\bigcup_{i\in I} A_i, *\right)$ with the operation $*$ defined as

$$a * b = (1 - \varphi_{i,i}(a)) + \varphi_{i,j}(b) + c_{i,j}, \qquad \text{for } a \in A_i \text{ and } b \in A_j.$$

Proposition (P.J., A.P., D.S., A.Z.)

The sum of an indecomposable affine mesh over a set I is a medial quandle with orbits equal to A_i, $i \in I$.
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3-element medial quandles

Example

Medial quandles of size 3

1. $(\mathbb{Z}_3;\ 2;\ 0)$

	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

2. $(\mathbb{Z}_2, \mathbb{Z}_1;\ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right);\ \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right))$

	a	b	c
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3. $(\mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1;\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix})$

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c	c	c	c

3. $(\mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1;\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix})$

	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

3-element medial quandles

Example

Medial quandles of size 3

1. $(\mathbb{Z}_3;\ 2;\ 0)$

	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

2. $\left(\mathbb{Z}_2, \mathbb{Z}_1;\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)$

	a	b	c
a	a	a	b
b	b	b	a
c	c	c	c

3. $\left(\mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1;\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$

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Reductivity

Definition

A groupoid Q is called *m-reductive* if it satisfies

$$\underbrace{x\cdot(x\cdots(x}_{m\times}\cdot y)\cdots)=x$$

Fact

A quandle $\mathrm{Aff}(A,f)$ *is m-reductive if and only if* $(1-f)^m=0$.

Example

$\mathrm{Aff}(\mathbb{Z}_{p^m},1-p)$ is m-reductive but not $m-1$-reductive.

Proposition (P.J.,A.P.,A.Z.)

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2-reductive quandles

Proposition

Let Q be a medial quandle Then TFAE

1. *Q is 2-reductive,*
2. *every orbit of Q is a left-zero band,*
3. $\mathrm{LMlt}(Q)$ *is commutative.*

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If one of the orbits of Q has one element then Q is 2-reductive.

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The number of non-isomorphic 2-reductive medial quandles of size n is $2^{\frac{1}{4}n^2+o(n^2)}$.

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Numbers of medial quandles

size	2-red.	other
1	1	0
2	1	0
3	2	1
4	5	1
5	15	3
6	55	3
7	246	5
8	1,398	12

size	2-reductive	other
9	10,301	10
10	98,532	45
11	1,246,479	9
12	20,837,171	268
13	466,087,624	11
14	13,943,041,873	?
15	563,753,074,915	36
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Conjecture

For each n, the number of 2-reductive medial quandles is bigger than the number of other medial quandles.

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An algebra $\mathcal{A}$ is called *subdirectly irreducible* if there exists on $\mathcal{A}$ a unique minimal non-trivial congruence, called *the monolith*.

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Proposition (P.J., A.P., A.Z.)

Let Q be a finite subdirectly irreducible medial quandle. Then Q is either a quasigroup or reductive.

Fact

A subdirectly irreducible medial idempotent quasigroup is polynomially equivalent to a subdirectly irreducible $\mathbb{Z}[x, x^{-1}, (1-x)^{-1}]$*-module.*

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The only finite simple reductive medial quandle is the two-element left zero band.

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Subdirectly irreducible reductive quandles

Theorem (P.J.,A.P.,A.Z.)

Let Q be a non-connected m-reductive medial quandle. Then Q is subdirectly irreducible if and only if it is isomorphic to the sum of the affine mesh

$$((A, \underbrace{\varphi(A), \varphi(A), \ldots}_{n-times});$$

$$\begin{pmatrix} \varphi & \varphi^2 & \varphi^2 & \ldots & \varphi^2 \\ 1 & \varphi & \varphi & \ldots & \varphi \\ 1 & \varphi & \varphi & \ldots & \varphi \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \varphi & \varphi & \ldots & \varphi \end{pmatrix}; \begin{pmatrix} 0 & -\varphi(c_{2,1}) & \ldots & -\varphi(c_{i,1}) & \ldots & -\varphi(c_{j,1}) \\ c_{2,1} & 0 & \ldots & \varphi(c_{2,1}-c_{i,1}) & \ldots & \varphi(c_{2,1}-c_{j,1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{i,1} & \varphi(c_{i,1}-c_{2,1}) & \ldots & 0 & \ldots & \varphi(c_{j,1}-c_{i,1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{j,1} & \varphi(c_{j,1}-c_{2,1}) & \ldots & \varphi(c_{j,1}-c_{i,1}) & \ldots & 0 \end{pmatrix}),$$

where

1. *A is a subdirectly irreducible* $\mathbb{Z}[x]/(1-x)^{m-1}$*-module,*
2. $\varphi = 1 - x$,
3. $0 < n \leqslant k$, *where* $k = |A/\varphi(A)|$,
4. $c_{i,1} - c_{j,1} \notin \varphi(A)$, *for each* $1 < i \neq j \in I$,
5. *A is generated by the set* $\varphi(A) \cup \{c_{i,1} \mid i \in I\}$.