Structure of medial quandles

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Definition of quandles

Definition

A groupoid Q is called a quandle if it satisfies

- L_x is an endomorphism, for each $x \in Q$, (left distributivity)
- L_x is a permutation, for each $x \in Q$, (left quasigroup)
- x is a fixed point of L_x, for each $x \in Q$. (idempotency)

Theorem (D. Joyce)

The knot quandle *is a classifying invariant of knots.*

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Examples of quandles

Example (Right zero band)

The groupoid $(Q, *)$ with the operation $x * y = y$.

Example (Group conjugation)

Let $(G, \cdot)$ be a group and let $a * b = a \cdot b \cdot a^{-1}$.

Example (Galkin's representation)

Let G be a group and $H \leqslant G$. Let f be an automorphism of G with $H \leqslant C_G(f)$. Let Q be the set of cosets $\{aH;\ a \in G\}$. We define

$$aH * bH = a \cdot f(a^{-1} \cdot b) \cdot H.$$

Then $(Q, *)$ is a quandle denoted by $\mathrm{Gal}(G, H, f)$.

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Permutation groups

Definitions

- The *left multiplication group* of Q is the permutation group $\mathrm{LMlt}(Q) = \langle L_x;\ x \in Q \rangle$.
- The *displacement group* of Q is the permutation group $\mathrm{Dis}(Q) = \langle L_x L_y^{-1};\ x, y \in Q \rangle$.

Proposition

- $\mathrm{Dis}(Q) \trianglelefteq \mathrm{LMlt}(Q)$,
- *the group* $\mathrm{LMlt}(Q)\,/\,\mathrm{Dis}(Q)$ *is cyclic,*
- *the natural actions of* $\mathrm{LMlt}(Q)$ *and* $\mathrm{Dis}(Q)$ *on* Q *have the same orbits.*

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Galkin's representation of orbits

Proposition (A. Hulpke, D. Stanovský, P. Vojtěchovský)

Let Q be a quandle, and let $e \in Q$. Then $\mathrm{Gal}(\mathrm{Dis}(Q), \mathrm{Dis}(Q)_e, (\cdot)^{L_e})$ *is well defined and isomorphic to the orbit e^Q.*

Medial quandles

Definition

A groupoid is called *medial*, if it satisfies

$$(x * y) * (u * z) = (x * u) * (y * z)$$

Definition

Let $(A, +)$ be an abelian group and $f \in \mathrm{Aut}(A)$. The set A with the operation

$$x * y = (1 - f)(x) + f(y)$$

forms a quandle called *affine* and denoted by $\mathrm{Aff}(A, f)$.

Observation

A quandle Q is affine if and only if it admits a Galkin's representation of form $\mathrm{Gal}(G, H, f)$ where G is abelian.

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Orbits of medial quandles

Proposition

A quandle is medial if and only if $\mathrm{Dis}(Q)$ *is abelian.*

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Every orbit of a medial quandle is affine.

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Sums of affine meshes

Definition

The *sum* of an indecomposable affine mesh $\mathcal{A} = (A_i, \phi_{i,j}, c_{i,j})$ over a set I is the groupoid $\left(\bigcup_{i\in I} A_i, *\right)$ with the operation $*$ defined as

$$a * b = \phi_{i,j}(a) + (1 - \phi_{j,j})(b) + c_{i,j}, \qquad \text{for } a \in A_i \text{ and } b \in A_j.$$

Proposition (P. J., A. P., D. S., A. Z.-D.)

The sum of an indecomposable affine mesh over a set I is a medial quandle with orbits equal to A_i, $i \in I$.
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Affine mesh

Definition

An *indecomposable affine mesh* is an n-tuplet of abelian groups $A_1, \ldots, A_n$, together with homomorphisms $\phi_{i,j} : A_i \to A_j$ and constants $c_{i,j} \in A_j$, for $i, j \in [1, \cdots, n]$, satisfying

(M1) $(1 - \phi_{i,i}) \in \mathrm{Aut}(A_i)$;

(M2) $c_{i,i} = 0$;

(M3) $\phi_{j,k} \circ \phi_{i,j} = \phi_{j',k} \circ \phi_{i,j'}$;

(M4) $\phi_{j,k}(c_{i,j}) = \phi_{k,k}(c_{i,k} - c_{j,k})$;

(M5) $A_j = \left\langle \bigcup_{i \in I} \left(c_{i,j} + \mathrm{Im}(\phi_{i,j})\right) \right\rangle$.

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3-element medial quandles

Example

Medial quandles of size 3

1. $(\mathbb{Z}_3;\ 2;\ 0)$

	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

2. $\left(\mathbb{Z}_2, \mathbb{Z}_1;\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)$

	a	b	c
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3. $\left(\mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1;\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$

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Homology of meshes

Definition

Two affine meshes $((A_i)_{i\in I}, (\phi_{i,j})_{i,j\in I}, (c_{i,j})_{i,j\in I})$ and $((A'_i)_{i\in I}, (\phi'_{i,j})_{i,j\in I}, (c'_{i,j})_{i,j\in I})$ are said to be *homologous* if

- there exists $\pi \in S_I$ such that $A_i \cong A'_{\pi i}$;
- for each $i \in I$, there exists ψ_i, an isomorphism between A_i and $A'_{\pi i}$, such that
$$\psi_j \circ \phi_{i,j} = \phi'_{\pi i,\pi j} \circ \psi_i; \tag{H1}$$
- for each $j \in I$, there exists $d_j \in A'_{\pi j}$ such that
$$\psi_j(c_{i,j}) = c'_{\pi i,\pi j} + \phi'_{\pi i,\pi j}(d_i) - \phi'_{\pi j,\pi j}(d_j). \tag{H2}$$

Proposition (P. J., A. P., D. S., A. Z.-D.)

The sums of two indecomposable affine meshes are isomorphic if and only if the meshes are homologous.

Homology group of affine meshes

Theorem (Burnside's orbit counting lemma)

Let a finite group G act on a set Ω. Let $\sim$ be an equivalence on g satisfying $g \sim h \Rightarrow \mathrm{fix}(g) = \mathrm{fix}(h)$. Then the action of G on Ω has

$$\frac{1}{|G|} \cdot \sum_{g \in G} \mathrm{fix}(g) = \frac{1}{|G|} \cdot \sum_{g \in R} \left|[g]_\sim\right| \cdot \mathrm{fix}(g)$$

orbits, where R is a transversal of $\sim$.

Fix abelian groups $\underbrace{A_1, \dots, A_1}_{n_1 \times}, \underbrace{A_2, \dots, A_2}_{n_2 \times}, \dots, \underbrace{A_m, \dots, A_m}_{n_m \times}$.

Permutations for homologies belong to $\prod_{i=1}^m S_{n_i}$.

The acting group is $G = \prod_{i=1}^{m} (A_i \rtimes \mathrm{Aut}(A_i)) \wr S_{n_i}$ acting on the set of all irreducible affine meshes.

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Reductivity

Definition

A groupoid Q is called *m-reductive* if it satisfies

$$(\cdots((x\underbrace{y)y)\cdots)y}_{m\times} = y$$

Fact

A quandle $\mathrm{Aff}(A, f)$ *is m-reductive if and only if* $(1-f)^m = 0$.

Example

$\mathrm{Aff}(\mathbb{Z}_{p^m}, 1-p)$ is m-reductive but not $m-1$-reductive.

Theorem (P. J., A. P., D. S., A. Z.-D.)

A medial quandle is m-reductive and not $m-1$*-reductive if and only if* $\mathrm{LMlt}(Q)$ *is nilpotent of degree* $m-1$.

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Decomposition of finite medial quandles

Theorem (K. Kearnes)

Let G be a finite subdirectly irreducible idempotent medial groupoid. Then G is strongly solvable or affine.

Conjecture

A medial quandle Q is strongly solvable if and only if Q is reductive.

Conjecture

Let Q be a finite medial quandle. Then $Q = L \times R$, where R is reductive and L is a quasigroup.

Theorem

Every finite medial quasigroup quandle is polynomially equivalent to a module over $\mathbb{Z}[x]/(x^n + x^{n-1} + \cdots + x + 1)$.

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2-reductive quandles

Theorem

Let Q be the sum of an irreducible affine mesh $(A_i, \phi_{i,j}, c_{i,j})$. Then TFAE

1. *Q is 2-reductive,*
2. *every orbit of Q is a right-zero band,*
3. *$\phi_{i,j} = 0$, for every $i, j \in I$,*
4. *$\mathrm{LMlt}(Q)$ is commutative.*

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If one of the orbits of Q has one element then Q is 2-reductive.

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2-reductive affine meshes

Lemma

$(A_i, \phi_{i,j}, c_{i,j})$ is an irreducible affine mesh whose sum is a 2-reductive mesh if and only if

- $\phi_{i,j} = 0$, *for each* $i, j \in I$,
- $c_{i,i} = 0$, *for each* $i \in I$,
- *for each* $j \in I$, $A_j = \langle c_{i,j},\ i \in I \rangle$.

Theorem (P. J., A. P., D. S., A. Z.-D.)

The number of 2-reductive medial quandles of size n is

$$2^{\frac{1}{4}n^2 + \varepsilon(n)},$$

for some function $\varepsilon(n)$ with $O(n \log n) < |\varepsilon(n)| < o(n^2)$.

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Numbers of medial quandles

size	2-red.	other
1	1	0
2	1	0
3	2	1
4	5	1
5	15	3
6	55	3
7	246	5
8	1,398	12

size	2-reductive	other
9	10,301	10
10	98,532	45
11	1,246,479	9
12	20,837,171	268
13	466,087,624	11
14	13,943,041,873	?
15	563,753,074,915	36
16	30,784,745,506,212	?

Conjecture

For each n, the number of 2-reductive medial quandles is bigger than the number of other medial quandles.

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