Free medial quandles

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Definition of quandles

Definition

A groupoid $(Q, *)$ is called a *quandle*, if it satisfies

- $x * x = x$, *(idempotency)*
- $x * (y * z) = (x * y) * (x * z)$, *(left distributivity)*
- $\forall x, z \; \exists! y; \quad x * y = z$. *(left quasigroup)*

Theorem (D. Joyce)

The knot quandle is a classifying invariant of knots.

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Examples of quandles

Example (Right zero band)

The groupoid $(Q,*)$ with the operation $x*y=y$.

Example (Group conjugation)

Let $(G,\cdot)$ be a group and let $a*b=a\cdot b\cdot a^{-1}$.

Definition

Let $(A,+)$ be an abelian group and $f\in \mathrm{Aut}(A)$. The set A with the operation

$$x*y=(1-f)(x)+f(y)$$

forms a quandle called *affine* and denoted by $\mathrm{Aff}(A,f)$.

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A groupoid Q is called *medial* if it satisfies
$$(x * y) * (u * z) = (x * u) * (y * z)$$
and *involutory* if it satisfies
$$x * (x * y) = y.$$

Theorem (D. Joyce)

Let $n \in \mathbb{N}$ and $Q = \mathrm{Aff}(\mathbb{Z}^n, -1)$. Let
$$F = \{u \in Q;\ \text{at most one coordinate of } u \text{ is odd}\}.$$
Then F is a subquandle of Q which is a free $n+1$ generated involutory medial quandle over $(0, \ldots, 0), (1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$.

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Left translations

Definition

Let $(Q, *)$ be a groupoid. The mapping $L_x : a \mapsto x * a$ is called the *left translation* by x.

Definition

A groupoid Q is called a quandle if it satisfies

- L_x is an endomorphism, for each $x \in Q$, (left distributivity)
- L_x is a permutation, for each $x \in Q$, (left quasigroup)
- x is a fixed point of L_x, for each $x \in Q$. (idempotency)

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Permutation groups

Definitions

- The *left multiplication group* of Q is the permutation group $\mathrm{LMlt}(Q) = \langle L_x;\ x \in Q\rangle$.
- The *displacement group* of Q is the permutation group $\mathrm{Dis}(Q) = \langle L_x L_y^{-1};\ x, y \in Q\rangle$.

Proposition

- $\mathrm{LMlt}(Q)' \trianglelefteq \mathrm{Dis}(Q) \trianglelefteq \mathrm{LMlt}(Q)$,
- *the group* $\mathrm{LMlt}(Q)/\mathrm{Dis}(Q)$ *is cyclic,*
- *the natural actions of* $\mathrm{LMlt}(Q)$ *and* $\mathrm{Dis}(Q)$ *on* Q *have the same orbits.*

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Proposition (P.J., A.P., D.S., A.Z.-D.)

A quandle is medial if and only if $\mathrm{Dis}(Q)$ *is abelian. Moreover, in such a case* $\mathrm{Dis}(Q)$ *can be naturally endowed with a structure of a* $\mathbb{Z}[x, x^{-1}]$*-module.*

Proposition (P.J., A.P., D.S., A.Z.-D.)

Every orbit Qe of a medial quandle is affine of form $(\mathrm{Dis}(Q)/\mathrm{Dis}(Q)_e, x)$*, for any e in Qe.*

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Free medial quandles

Theorem (P.J., A.P., D.S., A.Z.-D.)

Let Q be a medial quandle generated by a subset X. Then Q is free over X if and only if, for each $e \in Q$,

- $|Qe \cap X| = 1$,
- *the action of* $\mathrm{Dis}(Q)$ *on* Qe *is free,*
- $\mathrm{Dis}(Q)$ *is a free* $\mathbb{Z}[x, x^{-1}]$*-module of rank* $|X| - 1$.

Construction of free medial quandles

$$1_i = \begin{cases} (\underbrace{0, 0, \dots, 0, 1}_{i\times}, 0, \dots, 0), & \text{for } i > 0, \\ (0, \dots, 0), & \text{for } i = 0 \end{cases}$$

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Let $n \in \mathbb{N}$ and let $Q = \mathrm{Aff}(\mathbb{Z}[x, x^{-1}]^n, x)$. Let

$$F = \{(f_i)_{1 \leqslant i \leqslant n} \in Q;\ \exists 0 \leqslant j \leqslant n;\ (f_i)_{1 \leqslant i \leqslant n} \equiv 1_j \pmod{(x-1)}\}.$$

Then F is a free medial quandle over $\{1_i; 0 \leqslant i \leqslant n\}$.

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Symmetric quandles

Definition

A quandle Q is called *m-symmetric*, for some $n \in \mathbb{N}$, if $L_e^m = 1$, for each $e \in Q$, i.e., if it satisfies the identity

$$\underbrace{x \cdot (x \cdots (x}_{m\times} \cdot y) \cdots) = y.$$

Proposition (P.J., A.P., D.S., A.Z.-D.)

A medial quandle Q is m-symmetric if and only if $(x^{m-1} + x^{m-2} + \cdots + x + 1) \cdot \mathrm{Dis}(Q) = 0$. In this case $\mathrm{Dis}(Q)$ is a $\mathbb{Z}[x]/(x^{m-1} + x^{m-2} + \cdots + x + 1)$-module.

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Theorem (P.J., A.P., D.S., A.Z.-D.)

Let $n, m \in \mathbb{N}$ and let $Q = \mathrm{Aff}(\mathbb{Z}[x]/(x^{m-1} + x^{m-2} + \cdots + x + 1), x)$. Let

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Example

Let $m = 2$. Then $\mathbb{Z}[x]/(x^{m-1} + x^{m-2} + \cdots + x + 1) \cong \mathbb{Z}$ and $x - 1 \equiv 2 \pmod{(x+1)}$.

Example

Let $n = 2$. We know that $\mathbb{Z}[x]/(x^{m-1} + x^{m-2} + \cdots + x + 1) \cong \prod_{d|m,d>1} \mathbb{Z}[\zeta_d]$, where ζ_d is a d-th primitive root of 1 in $\mathbb{C}$. Hence the free two-generated m-symmetric medial quandle is the subquandle of $\prod \mathrm{Aff}(\mathbb{Z}[\zeta_d], \zeta_d)$ generated by $(0, \ldots, 0)$ and $(1, \ldots, 1)$.

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