Medial quandles

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Definition of quandles

Definition

A groupoid $(Q, *)$ is called a *quandle*, if it satisfies

- $x * x = x,$ *(idempotency)*
- $x * (y * z) = (x * y) * (x * z),$ *(left distributivity)*
- $\forall x, z \; \exists! y; \quad x * y = z.$ *(left quasigroup)*

Theorem (D. Joyce)

The knot quandle is a classifying invariant of knots.

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Examples of quandles

Example (Right zero band)

The groupoid $(Q,*)$ with the operation $x*y=y$.

Example (Group conjugation)

Let $(G,\cdot)$ be a group and let $a*b=a\cdot b\cdot a^{-1}$.

Example (Galkin's representation)

Let G be a group and $H\leqslant G$. Let f be an automorphism of G with $H\leqslant C_G(f)$. Let Q be the set of cosets $\{aH;\ a\in G\}$. We define

$$aH*bH=a\cdot f(a^{-1}\cdot b)\cdot H.$$

Then $(Q,*)$ is a quandle denoted by $\mathrm{Gal}(G,H,f)$.

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Not every quandle has a Galkin's representation

Example

Consider

Q	a	b	c
a	a	b	c
b	a	b	c
c	b	a	c

Then Q does not have a Galkin's representation.

Proof.

Let there exist G and $H < G$ with $[G : H] = 3$ and $f \in \mathrm{Aut}(G)$ with $H \leqslant C_G(f)$, a Galkin's representation of Q. Then $a * c = b * c = c * c = c$ and therefore

$$af(a^{-1})H = bf(b^{-1})H = cf(c^{-1})H = H.$$

This implies $a * x = b * x = c * x$, for each $x \in Q$, a contradiction. □

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Left translations

Definition

Let $(Q, *)$ be a groupoid. The mapping $L_x : a \mapsto x * a$ is called the *left translation* by x.

Definition

A groupoid Q is called a quandle if it satisfies

- L_x is an endomorphism, for each $x \in Q$, (left distributivity)
- L_x is a permutation, for each $x \in Q$, (left quasigroup)
- x is a fixed point of L_x, for each $x \in Q$. (idempotency)

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Permutation groups

Definitions

- The *left multiplication group* of Q is the permutation group $\mathrm{LMlt}(Q) = \langle L_x;\ x \in Q\rangle$.
- The *displacement group* of Q is the permutation group $\mathrm{Dis}(Q) = \langle L_x L_y^{-1};\ x, y \in Q\rangle$.

Proposition

Let Q be generated by $x_1, \ldots, x_n$. Then

- $\mathrm{LMlt}(Q)$ *is generated by* $L_{x_1}, \ldots, L_{x_n}$,
- $\mathrm{Dis}(Q)$ *is generated by* $(L_{x_i} L_{x_1}^{-1})^{L_{x_j}^j}$, *for* $1 \leqslant i, j \leqslant n$ *and* $j \in \mathbb{Z}$.

Example

Let $R = \mathrm{Gal}(\mathbb{Z}[x, x^{-1}], 1, x)$ and let Q be the subquandle of R generated by 0 and 1. Then $\mathrm{Dis}(Q) \simeq \mathbb{Z}^\omega$.

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Normality of the displacement group

Proposition

- $\mathrm{LMlt}(Q)' \trianglelefteq \mathrm{Dis}(Q) \trianglelefteq \mathrm{LMlt}(Q)$,
- *the group* $\mathrm{LMlt}(Q)/\mathrm{Dis}(Q)$ *is cyclic,*
- *the natural actions of* $\mathrm{LMlt}(Q)$ *and* $\mathrm{Dis}(Q)$ *on* Q *have the same orbits.*

Example

Consider

Q	a	b	c
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Then $|\mathrm{LMlt}(Q)| = |\mathrm{Dis}(Q)| = 2$ and $|\mathrm{LMlt}(Q)'| = 1$

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Galkin's representation of orbits

Proposition (A. Hulpke, D. Stanovský, P. Vojtěchovský)

Let Q be a quandle, and let $e \in Q$. Let $G \in \{\mathrm{LMlt}(Q), \mathrm{Dis}(Q)\}$ and let f be the restriction of the conjugation by L_e in $\mathrm{LMlt}(Q)$ in G. Then $\mathrm{Gal}(G, G_e, f)$ is well defined and isomorphic to the orbit e^Q.

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- one orbit is isomorphic to $\mathrm{Gal}(\{\mathrm{id}_Q, (a,b)\}, \{\mathrm{id}_Q\}, \mathrm{id})$,
- the other is isomorphic to $\mathrm{Gal}(\{\mathrm{id}_Q, (a,b)\}, \{\mathrm{id}_Q, (a,b)\}, \mathrm{id})$.

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Connected quandles

Definition

A quandle Q is called *(algebraically) connected* if $\mathrm{Dis}(Q)$ acts transitively on Q.

Proposition

If Q is connected then $\mathrm{Dis}(Q) = \mathrm{LMlt}(Q)'$.

Theorem (A. Hulpke, D. Stanovský, P. Vojtěchovský)

Let Q be a connected quandle. Then

1. *$Q \cong \mathrm{Gal}(G, H, f)$ where $G = \mathrm{LMlt}(Q)'$, $e \in Q$, $H = G_e$ and f is the conjugation by L_e.*
2. *If $Q \cong \mathrm{Gal}(G, H, f)$ then $\mathrm{LMlt}(Q)'$ embeds into a quotient of G.*

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Enumerating small connected quandles

Searching for a permutation group G acting on $1, \dots, n$ such that

1. G' is transitive on $1, \dots, n$;
2.
3. there exists $\zeta \in Z(G_1)$ such that $\langle \zeta^G \rangle = G$.

Number of connected quandles of each size, up to isomorphism

n	1	2	3	4	5	6	7	8	9	10	11	12	13
$q(n)$	1	0	1	1	3	2	5	3	8	1	9	10	11

n	14	15	16	17	18	19	20	21	22	23	24
$q(n)$	0	7	9	15	12	17	19	9	0	21	42

n	25	26	27	28	29	30	31	32	33	34	35
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$$(x * y) * (u * z) = (x * u) * (y * z)$$

Definition

Let $(A, +)$ be an abelian group and $f \in \mathrm{Aut}(A)$. The set A with the operation

$$x * y = (1 - f)(x) + f(y)$$

forms a quandle called *affine* and denoted by $\mathrm{Aff}(A, f)$.

Observation

A quandle Q is affine if and only if it admits a Galkin's representation of form $\mathrm{Gal}(G, H, f)$ where G is abelian.

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Orbits of medial quandles

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A quandle is medial if and only if $\mathrm{Dis}(Q)$ *is abelian.*

Corollary

A connected quandle is medial if and only if it is affine.

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Every orbit of a medial quandle is affine.

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(M1) $(1 - \varphi_{i,i}) \in \mathrm{Aut}(A_i)$;

(M2) $c_{i,i} = 0$;

(M3) $\varphi_{j,k} \circ \varphi_{i,j} = \varphi_{j',k} \circ \varphi_{i,j'}$;

(M4) $\varphi_{j,k}(c_{i,j}) = \varphi_{k,k}(c_{i,k} - c_{j,k})$;

(M5) $A_j = \langle \bigcup_{i \in I} (c_{i,j} + \mathrm{Im}(\varphi_{i,j})) \rangle$.

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Sums of affine meshes

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The *sum* of an indecomposable affine mesh $\mathcal{A} = (A_i, \varphi_{i,j}, c_{i,j})$ over a set I is the groupoid $\left(\bigcup_{i \in I} A_i, *\right)$ with the operation $*$ defined as

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Proposition (P. J., A. Pilitowska, D. Stanovský, A. Zamojska-Dzienio)

The sum of an indecomposable affine mesh over a set I is a medial quandle with orbits equal to A_i, $i \in I$.
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3-element medial quandles

Example

Medial quandles of size 3

1. $(\mathbb{Z}_3;\ 2;\ 0)$

	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

2. $\left(\mathbb{Z}_2, \mathbb{Z}_1;\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)$

	a	b	c
a	a	b	c
b	a	b	c
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3. $\left(\mathbb{Z}_1, \mathbb{Z}_1, \mathbb{Z}_1;\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$

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3-element medial quandles

Example

Medial quandles of size 3

1. $(\mathbb{Z}_3;\ 2;\ 0)$

	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

2. $(\mathbb{Z}_2, \mathbb{Z}_1;\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})$

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Homology of meshes

Definition

Two affine meshes $((A_i)_{i\in I}, (\varphi_{i,j})_{i,j\in I}, (c_{i,j})_{i,j\in I})$ and $((A'_i)_{i\in I}, (\varphi'_{i,j})_{i,j\in I}, (c'_{i,j})_{i,j\in I})$ are said to be *homologous* if

- there exists $\pi \in S_I$ such that $A_i \simeq A'_{\pi i}$;
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Proposition (P. J., A. P., D. S., A. Z.-D.)

The sums of two indecomposable affine meshes are isomorphic if and only if the meshes are homologous.

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Examples of homologous meshes

Example

The meshes

$$\left(\mathbb{Z}_3, \mathbb{Z}_3;\ \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix};\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)$$

and

$$\left(\mathbb{Z}_3, \mathbb{Z}_3;\ \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix};\ \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\right)$$

are homologous through $\pi = 1$, $\psi_1 = \psi_2 = 1$, $d_1 = 1$ and $d_2 = 2$.

Homology group of affine meshes

Fix abelian groups $\underbrace{A_1,\dots,A_1}_{n_1\times},\underbrace{A_2,\dots,A_2}_{n_2\times},\dots,\underbrace{A_m,\dots,A_m}_{n_m\times}$.

Permutations for homologies belong to $\prod_{i=1}^{m} S_{n_i}$.

The acting group is $G=\prod_{i=1}^{m}\left(A_i\rtimes\mathrm{Aut}(A_i)\right)\wr S_{n_i}$ acting on the set of all irreducible affine meshes.

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Reductivity

Definition

A groupoid Q is called *m-reductive* if it satisfies

$$(\cdots((x\underbrace{y)y)\cdots)y}_{m\times} = y$$

Fact

A quandle $\mathrm{Aff}(A,f)$ is m-reductive if and only if $(1-f)^m = 0$.

Example

$\mathrm{Aff}(\mathbb{Z}_{p^m}, 1-p)$ is m-reductive but not $m-1$-reductive.

Conjecture

A medial quandle is m-reductive and not $m-1$-reductive if and only if $\mathrm{LMlt}(Q)$ is nilpotent of degree $m-1$.

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2-reductive quandles

Theorem

Let Q be the sum of an irreducible affine mesh $(A_i, \varphi_{i,j}, c_{i,j})$. Then TFAE

1. *Q is 2-reductive,*
2. *every orbit of Q is a right-zero band,*
3. *$\varphi_{i,i} = 0$, for every $i \in I$.*
4. *for every $j \in I$, there exists $i \in I$ such that $\varphi_{i,j} = 0$,*
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Corollary

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2-reductive affine meshes

Lemma

$(A_i, \varphi_{i,j}, c_{i,j})$ is an irreducible affine mesh whose sum is a 2-reductive mesh if and only if

- $\varphi_{i,j} = 0$, *for each* $i, j \in I$,
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- *for each* $j \in I$, $A_j = \langle c_{i,j},\ i \in I \rangle$.

Example

The number of non-isomorphic medial quandles with m orbits equal to $\mathbb{Z}_p$, is at least $p^{m^2 - 2m - (1+\log_p m)pm}$.

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Numbers of medial quandles

size	2-red.	other
1	1	0
2	1	0
3	2	1
4	5	1
5	15	3
6	55	3
7	246	5
8	1,398	12

size	2-reductive	other
9	10,301	10
10	98,532	45
11	1,246,479	9
12	20,837,171	268
13	466,087,624	11
14	13,943,041,873	?
15	563,753,074,915	36
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For each n, the number of 2-reductive medial quandles is bigger than the number of other medial quandles.

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