Equational theory of left divisible left distributive groupoids

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Definitions

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Left distributivity:

$$x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$$

Idempotency:

$$x \cdot x = x$$

Example

Let $(G, *)$ be a group and define

$$x \cdot y = x * y * x^{-1}$$

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Smallest non-GC example

Example (D. Larue; A. Drápal, T. Kepka, M. Musílek)

The following identity holds in GC:

$$(xy \cdot y) \cdot xx = xy \cdot (yx \cdot x)$$

while not in every LDI groupoid; the smallest counterexamples are

$\cdot$	1	2	3	4
1	1	2	4	3
2	1	2	4	3
3	4	4	3	1
4	3	3	1	4

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2	2	2	4	4
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Equivalence of idempotent varieties

Theorem (D. Joyce; T. Kepka; D. Larue)

The following varieties coincide

- *the variety generated by the groupoids of group conjugacy;*
- *the variety generated by the left distributive idempotent left quasigroups;*
- *the variety generated by the left cancellative left distributive idempotent groupoids;*
- *the variety generated by the left divisible left distributive idempotent groupoids.*

Left idempotency

Fact

Every LDLD groupoid satisfies the identity

$$x \cdot y = xx \cdot y$$

called the left (pseudo)-idempotency.

Proof.

For all x, y in G there exist $z \in G$ such that $x \cdot z = y$. Now

$$x \cdot y = x \cdot xz = xx \cdot xz = xx \cdot y.$$

□

Examples of non-idempotent LDLQ

Example

Define an operation ^ on $\mathbb{Z}$ as follows:

$$i \,\hat{}\, j = j + 1$$

Then $\mathbb{Z}(\hat{})$ is a left distributive left quasigroup.

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Let $(G, *)$ be a group and let $d \in G$. Define an operation ^ on G as follows:

$$a \,\hat{}\, b = d * b$$

Then $G(\hat{})$ is a left distributive left quasigroup.

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Equivalence of left idempotent varieties

Definition

Let $(G,*)$ be a group and X a subset of G. We define the *half-conjugacy* as the following binary operation on $G \times X$:

$$(a,x)\cdot(b,y) = (a*x*a^{-1}*b,y)$$

Theorem (T. Kepka; P. Dehornoy)

The following varieties coincide:

- *the variety generated by the groups with half-conjugacy;*
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Larue's identity in the non-idempotent case

Lemma

The identity

$$(xy \cdot y) \cdot xx = xy \cdot (yx \cdot x)$$

holds in

- *every LCLDLI groupoid;* *(D. Larue)*
- *every LDLD groupoid.* *(D. Stanovský)*

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LDLD with surjective squaring

Proposition (P.J.)

Let G be an LDLD groupoid with the mapping $a \mapsto a^2$ surjective. Then G lies in the variety generated by LCLDLI.

Lemma

Let G be a LDLI groupoid. Then

- *if G is left cancellative then $a \mapsto a^2$ is injective;*
- *if G is a left quasigroup then $a \mapsto a^2$ is bijective.*

Proof.

- If $a^2 = b^2$ then $a^2 = a^2 \cdot a = b^2 \cdot a = ba$. If $bb = ba$ then $b = a$.
- Choose $a \in G$. There exists $x \in G$ such that $ax = a$. Now $a^2 = (ax)^2 = ax^2$ giving $a = x^2$. □

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Idempotent congruence

Proposition (P.J.)

Let G be an LDLI groupoid and let ip_G be the smallest equivalence containing pairs (a, a^2), for all $a \in G$. Then

- *for all a, b, c in G, if $(a, b) \in \mathrm{ip}_G$ then $ac = bc$;*
- *ip_G is a congruence of G;*
- *every class of ip_G is a subgroupoid of G.*

Proposition (P.J.)

Let G be a LDLD groupoid. Then the following conditions are equivalent:

- *the homomorfism $a \mapsto a^2$ is onto;*
- *$\forall a \in G\ \exists x \in G$: $ax = a$ and $(a, x) \in \mathrm{ip}_G$;*
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Open questions

Question:

Does there exist an LDLD groupoid where $a \mapsto a^2$ is not surjective?
If yes, does it lie in the variety generated by LCLDLI?

Question:

Find an equational basis of the group conjugacy.
Could it be finite?

Question:

Find an equational basis of the group half-conjugacy.
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