Commutative automorphic p-loops

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AAA 80, Będlewo
2nd June 2010

CZECH UNIVERSITY OF LIFE SCIENCES PRAGUE

Quasigroups

Definition

Let $(G, \cdot)$ be a groupoid. The mapping $L_x : a \mapsto xa$ is called the *left translation* and the mapping $R_x : a \mapsto ax$ the right translation.

Definition (Combinatorial)

A groupoid $(Q, \cdot)$ is called a *quasigroup* if the mappings L_x and R_x are bijections, for each $x \in Q$.

Definition (Universal algebraic)

The algebra $(Q, \cdot, /, \backslash)$ is called a *quasigroup* if it satisfies the following identities:

$$x\backslash(x \cdot y) = y \qquad (x \cdot y)/y = x$$
$$x \cdot (x\backslash y) = y \qquad (x/y) \cdot y = x$$

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Loops

Definition

A quasigroup Q is called a *loop* if it contains the identity element.

Example (A minimal nonassociative loop)

	1	2	3	4	5
1	1	2	3	4	5
2	2	1	5	3	4
3	3	4	1	5	2
4	4	5	2	1	3
5	5	3	4	2	1

Multiplication Groups

Definitions

Let Q be a loop.

- The group generated by L_x and R_x, for all $x \in Q$, is called *the multiplication group* of Q and it is denoted by $\mathrm{Mlt}(Q)$.
- The subgroup of $\mathrm{Mlt}(Q)$ stabilizing the neutral element of Q is called *the inner mapping group* of Q and it is denoted by $\mathrm{Inn}(Q)$.

Fact

An inner mapping of a loop needs not to be an automorphism.

Definition

A loop Q is called an *automorphic loop* (or an *A-loop*) if $\mathrm{Inn}(Q) \leqslant \mathrm{Aut}(Q)$.

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Basic properties of A-loops

Fact

Any characteristic subloop of an A-loop is normal.

Theorem (R. H. Bruck, J. L. Paige)

Every monogenerated subloop of an A-loop is a group.

Notation

We write x^3 instead of $x \cdot (x \cdot x)$ or $(x \cdot x) \cdot x$.
We write x^{-1} instead of $1/x$ or $x\backslash 1$.

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Variety of A-loops

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Let Q be a loop. The inner mapping group of Q is generated by the mappings

$$L_{xy}^{-1}L_xL_y, \qquad R_{xy}^{-1}R_xR_y \quad \text{and} \quad L_x^{-1}R_x,$$

where $x, y \in Q$.

Corollary

A loop is an A-loop iff it satisfies the following three identities:

$$(xy)\backslash(x(y \cdot uv)) = ((xy)\backslash(x \cdot yu)) \cdot ((xy)\backslash(x \cdot yv)),$$
$$((uv \cdot x)y)/(xy) = ((ux \cdot y)/(xy)) \cdot ((vx \cdot y)/(xy)),$$
$$x\backslash(uv \cdot x) = (x\backslash(ux)) \cdot (x\backslash(vx)).$$

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Uniquely 2-divisible loops

Definition

A loop Q is called *uniquely 2-divisible* if the map $x \mapsto x^2$ is a bijection.

Lemma

Let Q be a finite commutative loop with both-sided inverses. Then Q is uniquely 2-divisible if and only if $|Q|$ is odd.

Proof.

"$\Rightarrow$": If Q is uniquely 2-divisible then it contains no element of order 2. Hence the bijection $x \mapsto x^{-1}$ has only one fixed point and the number of nonidentity elements of Q is even.

"$\Leftarrow$": Fix $c \in Q$. The set $\{(x,y);\ xy = c\}$ has size $|Q|$, that means an odd size. By commutativity, the set $\{(x,y);\ xy = c\ \&\ x \neq y\}$ is of an even size. Hence there exists $x \in Q$ such that $x^2 = c$. □

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Commutatives A-loops of odd orders

Proposition (P. J., M. K., P. V.)

Let $(Q, \cdot)$ be a uniquely 2-divisible commutative A-loop. We associate to Q an operation $\circ$ defined as:

$$x \circ y = \sqrt{(x \cdot y^2)/x^{-1}}$$

Then Q is a Bruck loop. Moreover, the powers in $(Q, \cdot)$ coincide with the powers in $(Q, \circ)$

Corollary

- *Lagrange theorem,*
- *If $p \mid |Q|$, for p prime, then there exists $x \in Q$ of order p,*
- *Existence of Sylow p-subloops,*
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Nilpotency of loops

Definition

Let Q be a loop. The *center* of Q is the set

$$Z(Q) = \{a \in Q;\ \varphi(a) = a\ \forall\varphi \in \mathrm{Inn}(Q)\}$$

Definition

Let Q be a loop. The *upper central series* of Q is

$$Z_0(Q) \leqslant Z_1(Q) \leqslant Z_2(Q) \leqslant \cdots \leqslant Z_n(Q) \leqslant \cdots \leqslant Q,$$

where $Z_0(Q) = \{1\}$ and $Z_i(Q)$ is the preimage of $Z\big(Q/Z_{i-1}(Q)\big)$. If there exists some n such that $Z_n(Q) = Q$ then Q is said to be *(centrally) nilpotent of class n*.

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Drápal's Construction

Theorem (A. Drápal, refined by P. Jedlička & D. Simon)

Let K be the q-element finite field, $\mathrm{char}(K) \neq 2$. Let k be an odd divisor either of $q-1$ or of $q+1$. Take ξ, a k-th primitive root of unity. We define an operation $*$ on the set $Q = K \times \mathbb{Z}_k$ as follows:

$$(a,i) * (b,j) = \left((a+b) \cdot \frac{(\xi^i + 1) \cdot (\xi^j + 1)}{2 \cdot (\xi^{i+j} + 1)} \, , \; i+j \right).$$

Then $(Q,*)$ is a commutative automorphic loop, $|Q|$ is odd and $Z(Q) = 1$.

p-loops

Definition

Let Q be a loop where each element generates a cyclic subgroup and let p be a prime. The loop is called a *p-loop* if, for each $x \in Q$, there exists k, such that $x^{p^k} = 1$.

Theorem (P. J., M. K., P. V.)

Let Q be a finite commutative automorphic loop and let p be a prime. Then Q is a p-loop if and only if $|Q| = p^k$ for some k.

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Nilpotency of commutative automorphic p-loops

Theorem (P. J., M. K., P. V.)

Let $Q(\cdot)$ be a uniquely 2-divisible commutative automorphic loop with associated Bruck loop $Q(\circ)$. Then, for each non-negative integer n,

$$Z_n(Q, \circ) = Z_n(Q, \cdot)$$

Corollary

Commutative automorphic p-loops are nilpotent, for each odd prime p.

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Commutative automorphic 2-loops with trivial center

Proposition (P. J., M. K., P. V.)

Let G be a vector space over $\mathbb{F}_2$ and let f be an automorphism of V. We construct an operation $*$ on $Q = V \times \mathbb{F}_2$ as follows:

$$(\vec{v}, i) * (\vec{w}, j) = (f^{i \cdot j}(\vec{v} + \vec{w}), i + j).$$

Then Q is a commutative automorphic loop of exponent 2. If f is identical then Q is a group, otherwise $Z(Q) = \{\vec{u} \in V;\ f(\vec{u}) = \vec{u}\} \times 0$.

Corollary

There exist commutative automorphic 2-loops with trivial center.

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Commutative automorphic loops of order p^3

Proposition (P. J., M. K., P. V.)

For $n \geqslant 1$ and $a, b \in \mathbb{Z}_n$, define $\mathcal{Q}_{a,b}(\mathbb{Z}_n)$ on $\mathbb{Z}_n^3$ as

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (x_1 + y_1 + (x_2 + y_2)x_3y_3 + \\ + a(x_2, y_2)_n + b(x_3, y_3)_n,\ x_2 + y_2,\ x_3 + y_3),$$

where

$$(x, y)_n = \begin{cases} 0 & \text{if } x + y < n, \\ 1 & \text{if } x + y \geqslant n. \end{cases}$$

The loop $\mathcal{Q}_{a,b}(\mathbb{Z}_n)$ is a commutative automorpic loop with $Z(\mathcal{Q}_{a,b}(\mathbb{Z}_n)) = \mathbb{Z}_n \times 0 \times 0$.

References

R. H. Bruck, J. L. Paige:
Loops whose inner mappings are automorphisms,
The Annals of Math., 2nd Series, **63**, no. 2, (1956), 308–323

A. Drápal: A class of comm. loops with metacyclic inner mapping groups, Comment. Math. Univ. Carolin. **49**,3 (2008) 357–382.

P. Jedlička, M. K. Kinyon, P. Vojtěchovský: Constructions of commutative automorphic loops, to appear in Comm. in Alg.

P. Jedlička, M. K. Kinyon, P. Vojtěchovský: Structure of commutative automorphic loops, to appear in Trans. of AMS

P. Jedlička, M. K. Kinyon, P. Vojtěchovský: Commutative automorphic loops of odd prime power order (preprint)

P. Jedlička, D. Simon: Commutative A-loops of order pq (preprint)

M. K. Kinyon, K. Kunen, J. D. Phillips: Every diassociative A-loop is Moufang, Proc. Amer. Math. Soc. **130** (2002), 619–624