An Invariant for Left Distributivity and Idempotency

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Left Distributivity and Idempotency

Definition

$$x \cdot yz = xy \cdot xz \qquad \text{left distributivity}$$

$$x = xx \qquad \text{idempotency}$$

Example

Let $(G, \cdot)$ be a group and let us define

$$x * y = x^{-1} \cdot y \cdot x.$$

Then $(G, *)$ is an LDI groupoid.

Theorem (D. Larue; A. Drápal, T. Kepka, M. Musílek)

Groups with conjugation generate a proper subvariety of LDI.

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The Weight of Terms

Definition

Let us have a real number w_x, for each variable x, and $p \in [0,1]$. The *weight* of a term t is defined inductively as

$$w(t) = \begin{cases} w_x & \text{for } t = x \\ p \cdot w(t_1) + (1-p) \cdot w(t_2) & \text{for } t = t_1 \cdot t_2 \end{cases}$$

Observation

Two LDI-equivalent terms have the same weight.

Theorem (S. Fajtłowicz, J. Mycielski)

Two terms have always the same weight if and only if they are equivalent modulo mediality ($xy \cdot zw = xz \cdot yw$) and idempotency.

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Terms as Trees

Definitions

An *address* is a word on $\{0, 1\}$. The empty address is denoted by $\varnothing$.

Let t be a term. The *subterm* of t at an address α is defined as

$$\mathrm{sub}(t, \alpha) = \begin{cases} t & \text{for } \alpha = \varnothing \\ \mathrm{sub}(t_1, \beta) & \text{for } t = t_1 \cdot t_2 \text{ and } \alpha = 0\beta \\ \mathrm{sub}(t_2, \beta) & \text{for } t = t_1 \cdot t_2 \text{ and } \alpha = 1\beta \end{cases}$$

An address α is *in* a term t if $\mathrm{sub}(t, \alpha)$ exists. The address α is called *external* if $\mathrm{sub}(t, \alpha)$ is a variable.

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Iterated Left Subterms

Definitions

For two terms t_1 and t_2, we denot by $t_1 \sqsubset t_2$ if $t_1 = \mathrm{sub}(t_2, 0^p)$, for some $p \geqslant 1$. The order $\sqsubseteq$ is defined analogously.

We write $t_1 \sqsubseteq_{\mathrm{LDI}} t_2$ if there exist $t'_1 \overset{\mathrm{LDI}}{=} t_1$ and $t'_2 \overset{\mathrm{LDI}}{=} t_2$ with $t'_1 \sqsubseteq t'_2$.

Observation

The relation $\sqsubseteq_{\mathrm{LDI}}$ is a pre-order. The equivalence associated with $\sqsubseteq_{\mathrm{LDI}}$ contains $\overset{\mathrm{LDI}}{=}$.

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$$xy \sqsubset xy \cdot x \sqsubset (xy \cdot x) \cdot (xy \cdot y)$$

Since $xy \overset{\mathrm{LDI}}{=} (xy \cdot x) \cdot (xy \cdot y)$ and $xy \overset{\mathrm{LDI}}{\neq} xy \cdot x$, the relation $\overset{\mathrm{LDI}}{=}$ is a proper subrelation of the equivalence given by $\sqsubseteq_{\mathrm{LDI}}$.

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Definition (P. Dehornoy)

The *cut* of a term t at an address α in t is defined as

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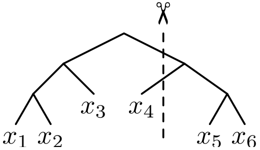

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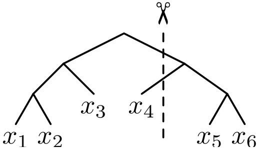

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Cuts and Left Iterated Terms

Observation

Let $s \sqsubseteq t$. Then s is a cut of t.

Proposition (P. Dehornoy)

Let s be a cut of a term t. Then $s \sqsubseteq_{\mathrm{LD}} t$.

Proposition (P.J.)

Let s be a cut of a term t. If t' is obtained from t in one step, then there exist two external addresses $\alpha \geqslant \beta$ in t' such that

$$s \stackrel{\mathrm{LDI}}{=} \mathrm{cut}(t', \alpha) \cdot \mathrm{cut}(t', \beta).$$

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Set of All Cuts

Definition

For a term t, we denote by $\mathrm{Cut}(t)$ the smallest set satisfying

- every cut of t belongs to $\mathrm{Cut}(t)$;
- if t' is LDI-equivalent to some $t \in \mathrm{Cut}(t)$ then $t' \in \mathrm{Cut}(t)$;
- if s and s' belong to $\mathrm{Cut}(t)$ and $s' \sqsubseteq s$ then $s \cdot s'$ belongs to $\mathrm{Cut}(t)$.

Theorem (P.J.)

The following conditions are equivalent for two terms s and t:

- $s \sqsubseteq_{\mathrm{LDI}} t$;
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An Example

Fact

$$x \cdot y \overset{\text{LDI}}{\neq} (x \cdot xy)(yx \cdot y)$$

Observation

$$(x \cdot xy)(yx \cdot y) \overset{\text{RD}}{=} (x \cdot xy)(yy \cdot xy) \overset{\text{RD}}{=} (x \cdot yy)(xy) \overset{\text{I}}{=} (xy)(xy) \overset{\text{I}}{=} xy$$

Lemma

Let us take the following weight: $w_x = 1, w_y = -1, p = 1/2$*. Then, for every* t *in* $\mathrm{Cut}(xy)$*, the weight of* t *is positive.*

Proof.

- $w(x) = 1, w(xy) = 0$;
- $w(t') = w(t)$ for $t' \overset{\text{LDI}}{=} t$;
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Proof.

- $w(x) = 1$, $w(xy) = 0$;
- $w(t') = w(t)$ for $t' \overset{\mathrm{LDI}}{=} t$;
- if $w(s) \geqslant 0$ and $w(s') \geqslant 0$ then $w(s \cdot s') \geqslant 0$. □

The Example

Proposition

$$x \cdot y \overset{\mathrm{LDI}}{\neq} (x \cdot xy)(yx \cdot y)$$

Proof.

The term $(x \cdot xy)y$ is a cut of $(x \cdot xy)(yx \cdot y)$.
Its weight is $-{}^1/_4$.
According to the previous lemma, $(x \cdot xy)y \notin \mathrm{Cut}(xy)$.
And hence $x \cdot y \overset{\mathrm{LDI}}{\neq} (x \cdot xy)(yx \cdot y)$. □

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Bibliography

P. Dehornoy:
Braids and Self-Distributivity
Progress in Mathematics **192**; Birkhäuser, 2000

A. Drápal, T. Kepka, M. Musílek:
Group Conjugation has Non-Trivial LD-Identities
Comment. Mathematicae Univ. Carolinae **35/2**, 1994, 596–606

P. Jedlička: On a Partial Syntactical Criterion for the Left Distributivity and the Idempotency
to appear in Mathematica Slovaca

D. Larue:
Left Distributive Idempotent Algebras
Communications in Algebra **27/5**, 1999, 2003–2009