Combinatorial Construction of the Weak Order of a Coxeter Group

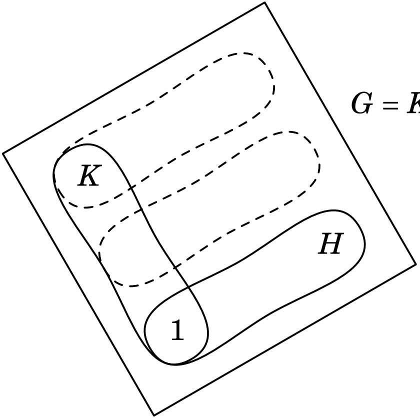
Přemysl Jedlička

Department of Mathematics
Faculty of Engineering (former Technical Faculty)
Czech University of Life Sciences (former Czech University of Agriculture), Prague

April 2008 – Denver, Colorado

ČESKÁ ZEMĚDĚLSKÁ UNIVERZITA V PRAZE

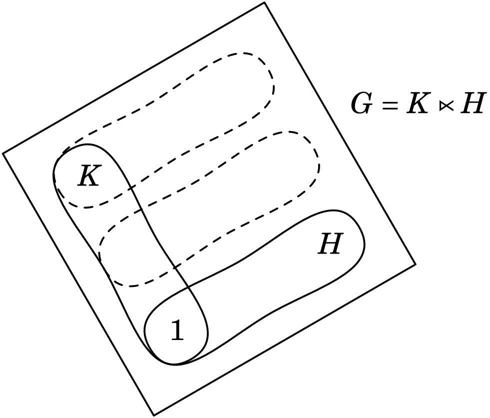
Semidirect Product of Groups



$G = K \ltimes H$

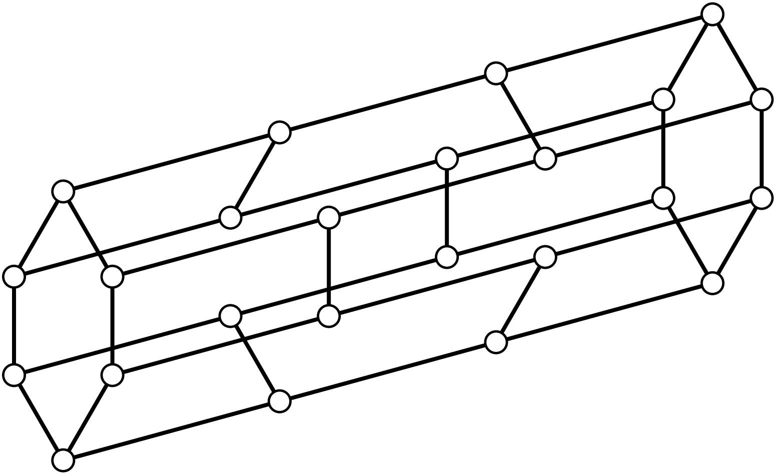
$$(k, h) * (k', h') = (k \cdot k', h \cdot \varphi_k(h'))$$

Semidirect Product of Groups

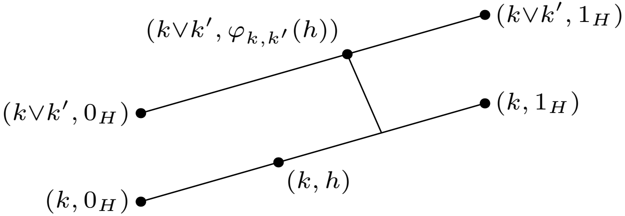


$$(k, h) * (k', h') = (k \cdot k', h \cdot \varphi_k(h'))$$

A Lattice That Is a Semidirect Product



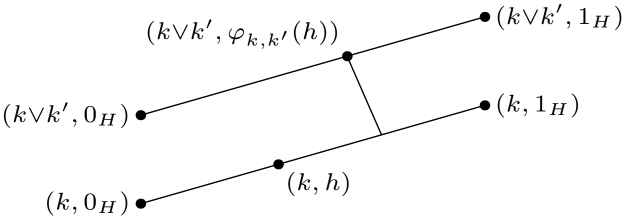
Definition of the Mappings φ and ψ



$$(k', 0_H) \vee (k, h) = (k \vee k', \varphi_{k,k'}(h))$$

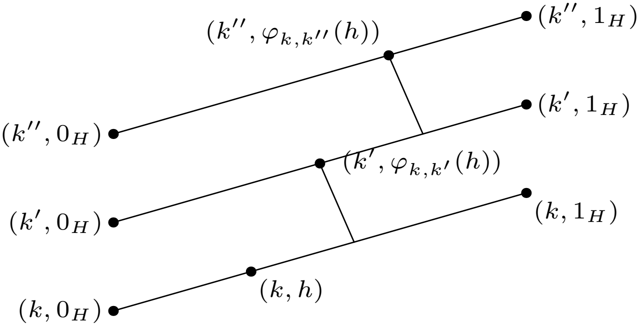
$$(k, 1_H) \wedge (k', h) = (k \wedge k', \psi_{k',k}(h))$$

Definition of the Mappings φ and ψ



$$(k', 0_H) \vee (k, h) = (k \vee k', \varphi_{k,k'}(h))$$
$$(k, 1_H) \wedge (k', h) = (k \wedge k', \psi_{k',k}(h))$$

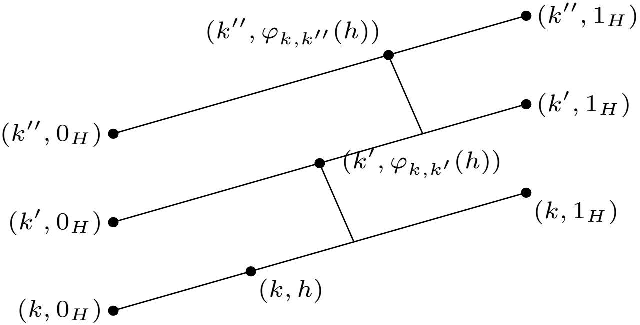
Properties of the Mapping φ



$\varphi_{k,k''} = \varphi_{k',k''} \circ \varphi_{k,k'}$ for $k \leqslant k' \leqslant k''$

$\varphi_{k,k'}(h \vee h') = \varphi_{k,k'}(h) \vee \varphi_{k,k'}(h')$

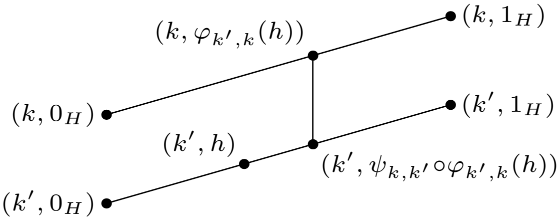
Properties of the Mapping φ



$$\varphi_{k,k''} = \varphi_{k',k''} \circ \varphi_{k,k'} \qquad \text{for } k \leqslant k' \leqslant k''$$

$$\varphi_{k,k'}(h \vee h') = \varphi_{k,k'}(h) \vee \varphi_{k,k'}(h')$$

Connection Between φ and ψ



$$\psi_{k,k'} \circ \varphi_{k',k}(h) \geqslant h$$

Semidirect Product of Lattices

Theorem (P.J.)

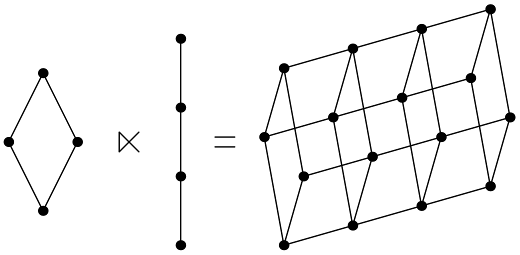
Let K, H be two lattices and let $\varphi, \psi : K^2 \to H^H$ be two mappings satisfying eight specific conditions. Then the set $K \times H$ with the operations $\vee$, $\wedge$, defined by

$$(k, h) \vee (k', h') = (k \vee k',\ \varphi_{k,k'}(h) \vee \varphi_{k',k}(h')),$$
$$(k, h) \wedge (k', h') = (k \wedge k',\ \psi_{k,k'}(h) \wedge \psi_{k',k'}(h')),$$

forms a lattice, called the semidirect product *of K and H and denoted by $K \ltimes_{\psi}^{\varphi} H$.*

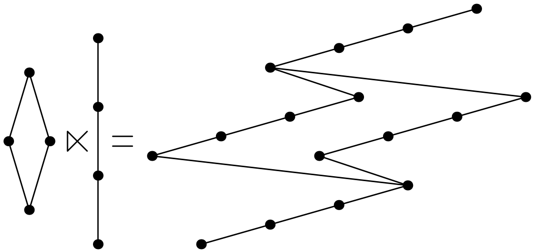
Example of a Semidirect Product 1

Let K, H be two arbitrary lattices and let $\varphi_{k,k'} = \psi_{k,k'} = \mathrm{id}_H$, for all k, k' in K. Then the semidirect product $K \ltimes_{\psi}^{\varphi} H$ is the carthesian product $K \times H$.



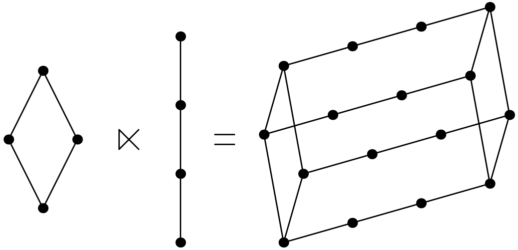
Example of a Semidirect Product 2

Let K, H be two arbitrary lattices and let, for each $k \leqslant k'$ and h in H, be $\varphi_{k,k'}(h) = 0_H$, $\psi_{k',k}(h) = 1_H$. Then we have $(k, h) \leqslant (k', h')$ if and only if $k < k'$ in K or $k = k'$ and $h \leqslant h'$ in H. Therefore, the semidirect product of K and H consists of $|K|$ copies of the lattice H arranged in the form of the lattice K.



Example of a Semidirect Product 3

Let K, H be arbitrary lattices and, for all $k \leqslant k'$, let $\varphi_{k,k'}(h) = 1_H$, for $h > 0_H$, and $\psi_{k',k}(h) = 0_H$, for $h < 1_H$. In this case if K has at least 2 elements and H has at least 3 elements, the lattice $K \ltimes_{\psi}^{\varphi} H$ is not modular.



Semidirect Product of Semilattices

Theorem (P.J.)

Let K and H be two meet-semilattices and let $\psi : K^2 \to \mathrm{End}(H)$ be a mapping satisfying, for all k, k' and k'' from K,

$$\psi_{k,k} = \mathrm{id}_H;$$
$$\psi_{k,k' \wedge k''} = \psi_{k \wedge k', k''} \circ \psi_{k,k'}.$$

Then the set $K \times H$ together with the operation $\wedge$ defined by

$$(k,h) \wedge (k',h') = (k \wedge k', \psi_{k,k'}(h) \wedge \psi_{k',k}(h'))$$

forms a semilattice. This semilattice is denoted $K \ltimes_\psi H$.

Coxeter Groups

Definition

A *Coxeter system* is a pair (W, S), where W is a group and S is a subset of W such that W has a presentation in form

$$W = \left\langle S;\ s^2 = 1,\ (st)^{m_{st}} = 1;\ \text{for all } s, t \in S \right\rangle$$

where $m_{st} \in \{2, 3, 4, \ldots, \infty\}$.
Such a group W is called a *Coxeter group*.

Examples

- Weyl groups
- Dihedral groups
- Groups of symmetries of a Euclidean space
- Groups of symmetries of an affine space

Coxeter Groups

Definition

A *Coxeter system* is a pair (W, S), where W is a group and S is a subset of W such that W has a presentation in form

$$W = \left\langle S;\ s^2 = 1,\ (st)^{m_{st}} = 1;\ \text{for all } s, t \in S \right\rangle$$

where $m_{st} \in \{2, 3, 4, \ldots, \infty\}$.
Such a group W is called a *Coxeter group*.

Examples

- Weyl groups
- Dihedral groups
- Groups of symmetries of a Euclidean space
- Groups of symmetries of an affine space

Weak Order

Definition

Let (W, S) be a Coxeter system and take $w \in W$. A *reduced expression* of w is an expression $w = s_1 s_2 \cdots s_k$, for $s_i \in S$, where k is minimal possible. The *lenght* of w, denoted by $\ell(w)$, is the lenght of this reduced expression.

Definition

Let (W, S) be a Coxeter system. We write $w \preccurlyeq w'$, for elements w and w' in W, if $\ell(w') = \ell(w) + \ell(w^{-1}w')$. This relation is called the *weak order* of W or sometimes the *weak Bruhat order* of W.

Weak Order

Definition

Let (W, S) be a Coxeter system and take $w \in W$. A *reduced expression* of w is an expression $w = s_1 s_2 \cdots s_k$, for $s_i \in S$, where k is minimal possible. The *lenght* of w, denoted by $\ell(w)$, is the lenght of this reduced expression.

Definition

Let (W, S) be a Coxeter system. We write $w \preccurlyeq w'$, for elements w and w' in W, if $\ell(w') = \ell(w) + \ell(w^{-1}w')$. This relation is called the *weak order* of W or sometimes the *weak Bruhat order* of W.

Properties of the Weak Order

Theorem (A. Björner)

The weak order on a Coxeter group W forms a meet-semilattice with 1 as the smallest element. The order forms a lattice if and only if W is finite.

Observation

Let (W, S) be a Coxeter system. The unoriented Hasse diagram of the weak order on W and the unlabelled Cayley graph of the presentation given by S are the same graphs.

Properties of the Weak Order

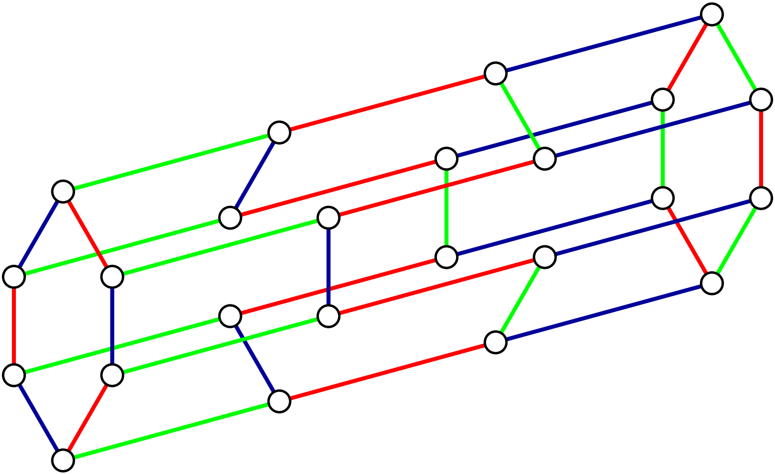
Theorem (A. Björner)

The weak order on a Coxeter group W forms a meet-semilattice with 1 as the smallest element. The order forms a lattice if and only if W is finite.

Observation

Let (W, S) be a Coxeter system. The unoriented Hasse diagram of the weak order on W and the unlabelled Cayley graph of the presentation given by S are the same graphs.

Weak Order of the Symmetric Group S_4



Standard Parabolic Subgroups

Definition

Let (W, S) be a Coxeter system and let X be a subset of S. The subgroup of W generated by X is called a *standard parabolic subgroup* and it is denoted by W_X.

Fact (well known)

Let (W, S) be a Coxeter system and let X be a subset of S. Then the pair (W_X, X) is a Coxeter system. For each element in W_X, the length in W_X and in W are the same.

Standard Parabolic Subgroups

Definition

Let (W, S) be a Coxeter system and let X be a subset of S. The subgroup of W generated by X is called a *standard parabolic subgroup* and it is denoted by W_X.

Fact (well known)

Let (W, S) be a Coxeter system and let X be a subset of S. Then the pair (W_X, X) is a Coxeter system. For each element in W_X, the length in W_X and in W are the same.

Reduced Elements

Definition

Let (W,S) be a Coxeter system and let X be a subset of S. An element w in W is called X-*reduced* if $x \not\leqslant w$, for all $x \in X$. The set of all X-reduced elements is denoted W^X

Proposition (V. Deodhar)

For each element w in W, there exists a unique decomposition $w = w_X w^X$ with $w_X \in W_X$ and $w^X \in W^X$.

Proposition (P.J.)

Let θ be this equivalence: $(w, w') \in \theta$ if and only if $w_X = w'_X$. Then θ is a congruence of the semilattice $(W, \leqslant)$ with $W/\theta \cong W_X$ and each of the congruence classes is isomorphic to W^X.

Reduced Elements

Definition

Let (W, S) be a Coxeter system and let X be a subset of S. An element w in W is called *X-reduced* if $x \not\leqslant w$, for all $x \in X$. The set of all X-reduced elements is denoted W^X

Proposition (V. Deodhar)

For each element w in W, there exists a unique decomposition $w = w_X w^X$ with $w_X \in W_X$ and $w^X \in W^X$.

Proposition (P.J.)

Let θ be this equivalence: $(w, w') \in \theta$ if and only if $w_X = w'_X$. Then θ is a congruence of the semilattice $(W, \leqslant)$ with $W/\theta \cong W_X$ and each of the congruence classes is isomorphic to W^X.

Reduced Elements

Definition

Let (W, S) be a Coxeter system and let X be a subset of S. An element w in W is called X*-reduced* if $x \not\leqslant w$, for all $x \in X$. The set of all X-reduced elements is denoted W^X

Proposition (V. Deodhar)

For each element w in W, there exists a unique decomposition $w = w_X w^X$ with $w_X \in W_X$ and $w^X \in W^X$.

Proposition (P.J.)

Let θ be this equivalence: $(w, w') \in \theta$ if and only if $w_X = w'_X$. Then θ is a congruence of the semilattice $(W, \leqslant)$ with $W/\theta \cong W_X$ and each of the congruence classes is isomorphic to W^X.

Construction for the Groups of Type A

Fact (well known)

The group of type A_n is the symmetric group on $n+1$ elements, with $s_i = (i, i+1)$. Let $X = \{s_1, \ldots, s_{n-1}\}$. Then

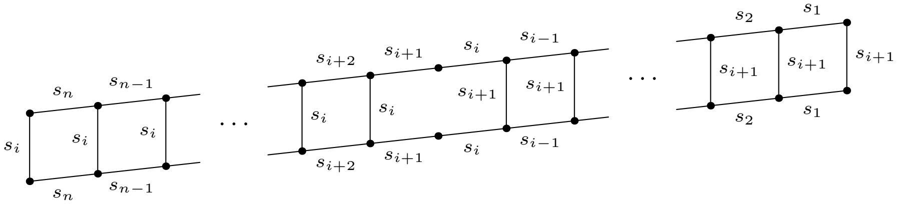
$$W_X = \{1,\ s_n,\ s_n s_{n-1},\ s_n s_{n-1} s_{n-2},\ \ldots\ ,\ s_n s_{n-1} \cdots s_2 s_1\}.$$

Construction for the Groups of Type A

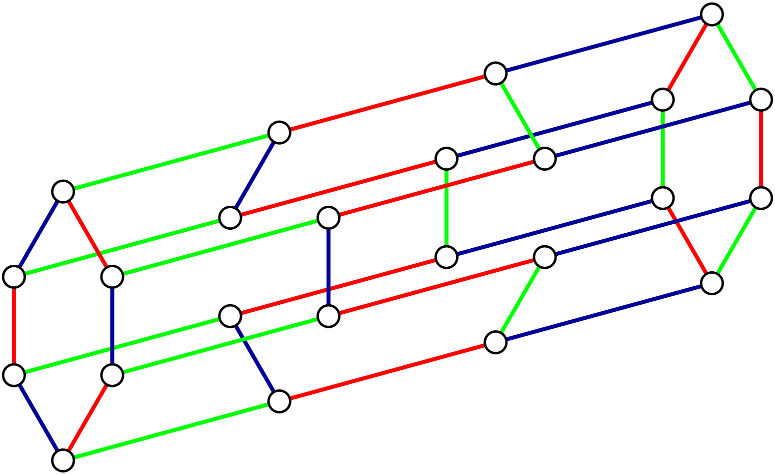
Fact (well known)

The group of type A_n is the symmetric group on $n+1$ elements, with $s_i = (i, i+1)$. Let $X = \{s_1, \ldots, s_{n-1}\}$. Then

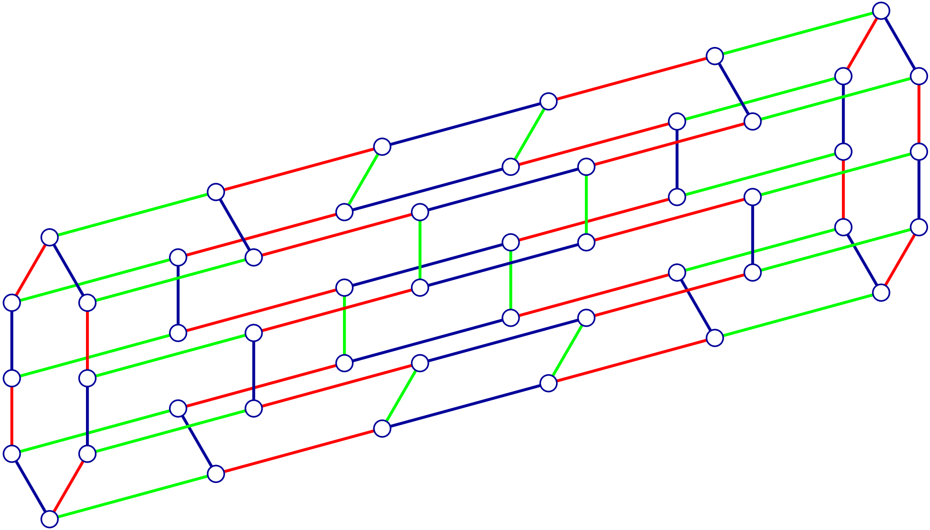
$$W_X = \{1,\ s_n,\ s_n s_{n-1},\ s_n s_{n-1} s_{n-2},\ \ldots\ ,\ s_n s_{n-1} \cdots s_2 s_1\}.$$



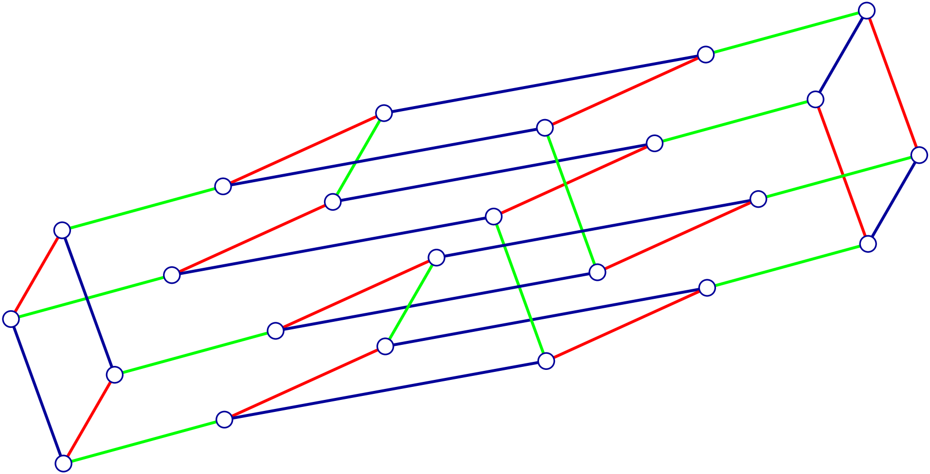
Cayley Graph / Weak Order of the Group of Type A_3



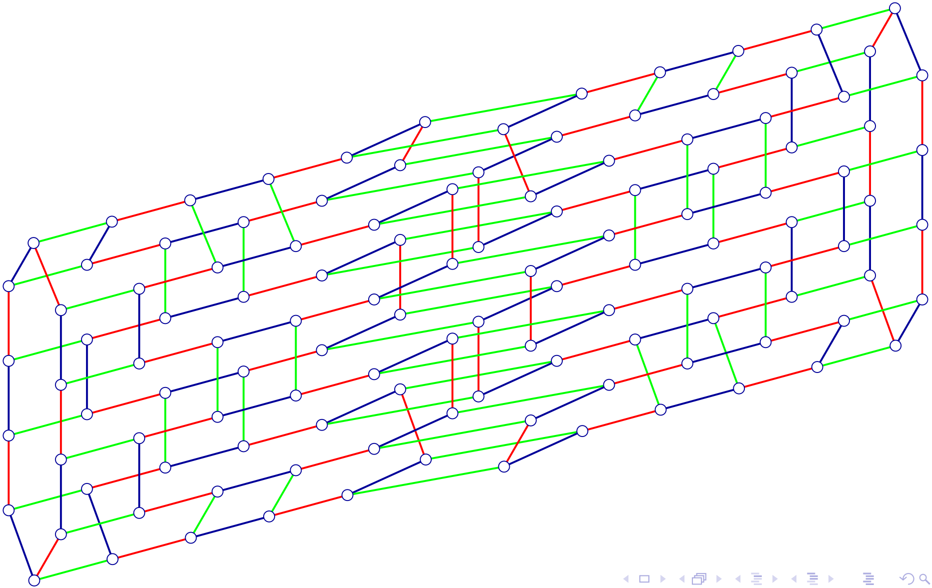
Cayley Graph / Weak Order of the Group of Type B_3



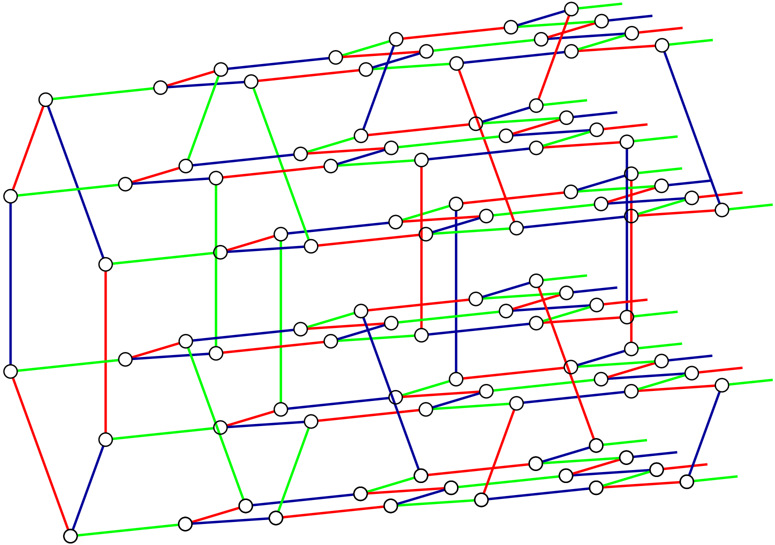
Cayley Graph / Weak Order of the Group of Type D_3



Cayley Graph / Weak Order of the Group of Type H_3



Cayley Graph / Weak Order of the Group of Type $\tilde{A}_2$



Bibliography

- A. Björner: Orderings of Coxeter Groups
  Cont. Math. 34, AMS: Providence, R.I., 1984, 175–195
- V. Deodhar:
  A splitting criterion for the Bruhat orderings on Coxeter groups
  Comm. in Algebra 15 (9), 1987, 1889–1894
- P. Jedlička: Combinatorial Construction of the Weak Order of a Coxeter Group
  Comm. in Algebra 33 (5), 2005, 1447–1460
- P. Jedlička: Semidirect Product of Lattices
  to appear in Algebra Universalis (2007)
- C. Le Conte de Poly-Barbut:
  Treillis de Cayley des groupes de Coxeter finis. Construction par récurrence et décompositions sur des quotiens.
  Math. Inf. Sci. Hum. 140, 1997, 11–33