Combinatorial Construction of the Weak Order of a Coxeter Group

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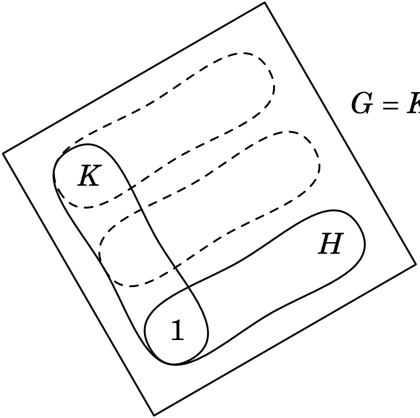
Semidirect Product of Groups



$G = K \ltimes H$

$$(k, h) * (k', h') = (k \cdot k', h \cdot \varphi_k(h'))$$

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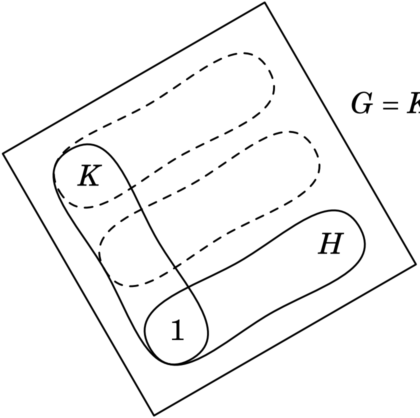

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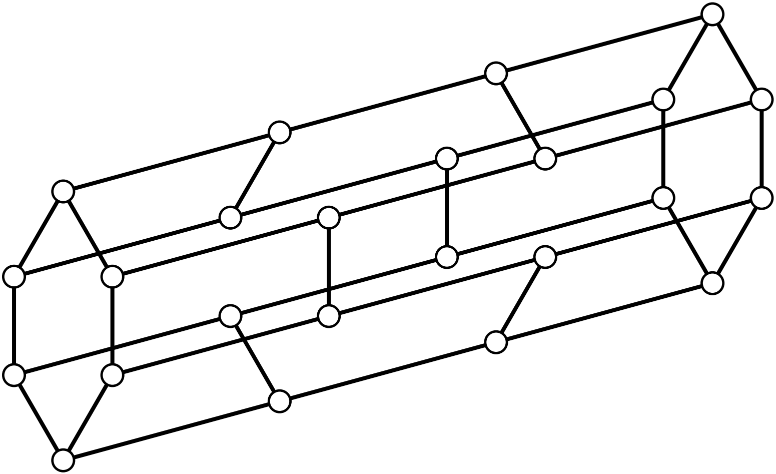
A Lattice That Is a Semidirect Product



Semidirect Product of Semilattices

Theorem (P.J.)

Let K and H be two join-semilattices and let $\varphi : K^2 \to \mathrm{End}(H)$ be a mapping satisfying, for all k, k' and k'' from K,

$$\varphi_{k,k} = \mathrm{id}_H;$$

$$\varphi_{k,k' \vee k''} = \varphi_{k \vee k', k''} \circ \varphi_{k,k'}.$$

Then the set $K \times H$ together with the operation $\vee$ defined by

$$(k,h) \vee (k',h') = (k \vee k', \varphi_{k,k'}(h) \vee \varphi_{k',k}(h'))$$

forms a semilattice.

Coxeter Groups

Definition

A *Coxeter system* is a pair (W, S), where W is a group and S is a subset of W such that W has a presentation in form

$$W = \left\langle S;\ s^2 = 1,\ (st)^{m_{st}} = 1;\ \text{for all } s, t \in S \right\rangle$$

where $m_{st} \in \{2, 3, 4, \ldots, \infty\}$.
Such a group W is called a *Coxeter group*.

Examples

- Weyl groups
- Dihedral groups
- Groups of symmetries of a Euclidean space
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Weak Order

Definition

Let (W, S) be a Coxeter system and take $w \in W$. A *reduced expression* of w is an expression $w = s_1 s_2 \cdots s_k$, for $s_i \in S$, where k is minimal possible. The *lenght* of w, denoted by $\ell(w)$, is the lenght of this reduced expression.

Definition

Let (W, S) be a Coxeter system. We write $w \leqslant w'$, for elements w and w' in W, if $\ell(w') = \ell(w) + \ell(w^{-1}w')$. This relation is called the *weak order* of W or sometimes the *weak Bruhat order* of W.

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Definition

Let (W,S) be a Coxeter system. We write $w \preccurlyeq w'$, for elements w and w' in W, if $\ell(w') = \ell(w) + \ell(w^{-1}w')$. This relation is called the *weak order* of W or sometimes the *weak Bruhat order* of W.

Properties of the Weak Order

Theorem (A. Björner)

The weak order on a Coxeter group W forms a meet-semilattice with 1 as the smallest element. The order forms a lattice if and only if W is finite.

Observation

Let (W, S) be a Coxeter system. The unoriented Hasse diagram of the weak order on W and the unlabelled Cayley graph of the presentation given by S are the same graphs.

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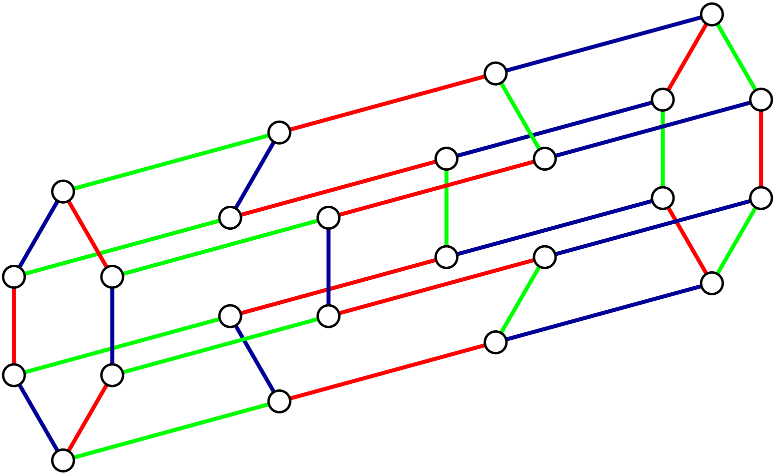
Weak Order of the Symmetric Group S_4



Standard Parabolic Subgroups

Definition

Let (W, S) be a Coxeter system and let X be a subset of S. The subgroup of W generated by X is called a *standard parabolic subgroup* and it is denoted by W_X.

Fact (well known)

Let (W, S) be a Coxeter system and let X be a subset of S. Then the pair (W_X, X) is a Coxeter system. For each element in W_X, the length in W_X and in W are the same.

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Reduced Elements

Definition

Let (W, S) be a Coxeter system and let X be a subset of S. An element w in W is called X*-reduced* if $x \not\leqslant w$, for all $x \in X$. The set of all X-reduced elements is denoted W^X

Proposition (V. Deodhar)

For each element w in W, there exists a unique decomposition $w = w_X w^X$ with $w_X \in W_X$ and $w^X \in W^X$.

Proposition (P.J.)

Let θ be this equivalence: $(w, w') \in \theta$ if and only if $w_X = w'_X$. Then θ is a congruence of the semilattice $(W, \leqslant)$ with $W/\theta \cong W_X$ and each of the congruence classes is isomorphic to W^X.

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Construction for the Groups of Type A

Fact (well known)

The group of type A_n is the symmetric group on $n+1$ elements, with $s_i = (i, i+1)$. Let $X = \{s_1, \ldots, s_{n-1}\}$. Then

$$W_X = \{1,\ s_n,\ s_n s_{n-1},\ s_n s_{n-1} s_{n-2},\ \ldots\ ,\ s_n s_{n-1} \cdots s_2 s_1\}.$$

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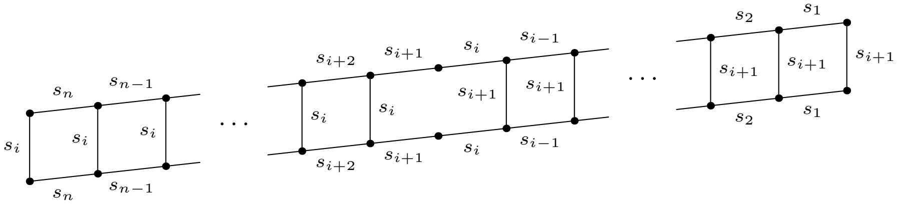
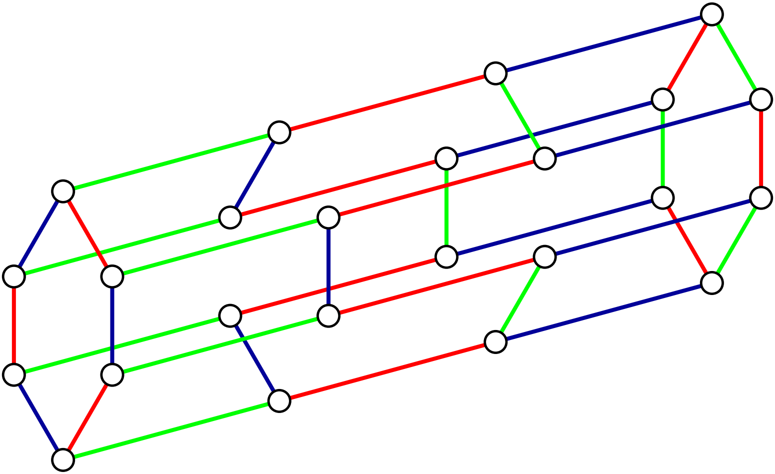
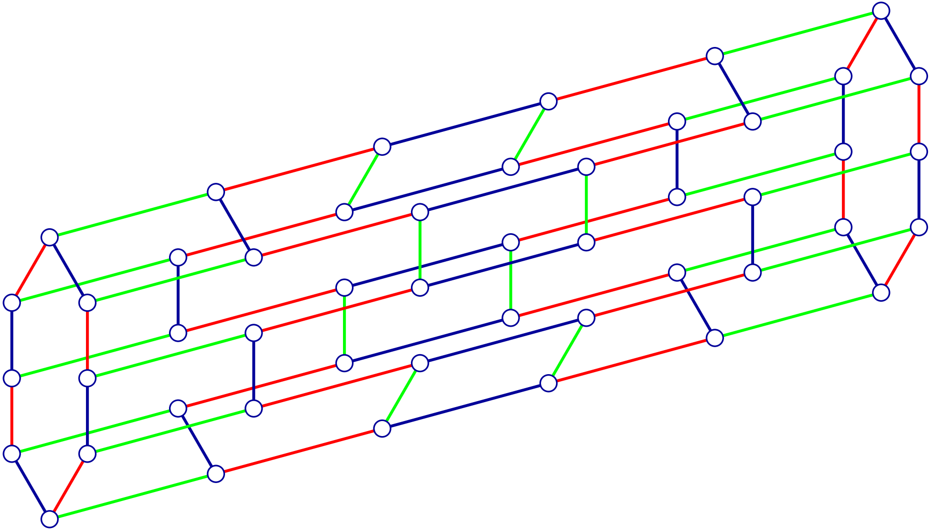


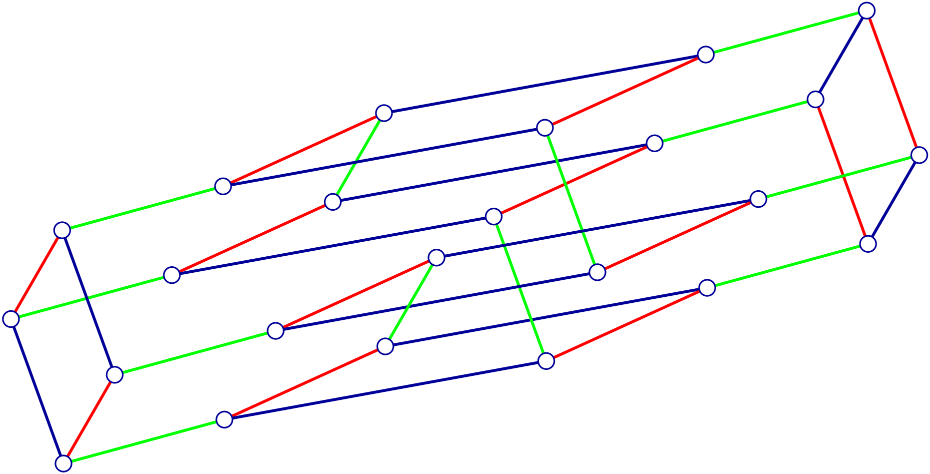
Cayley Graph / Weak Order of the Group of Type A_3



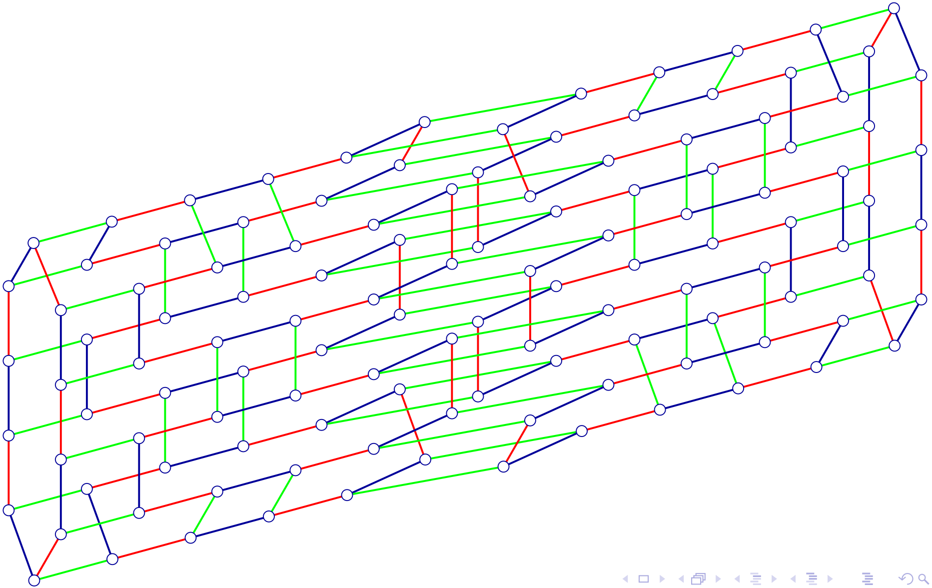
Cayley Graph / Weak Order of the Group of Type B_3



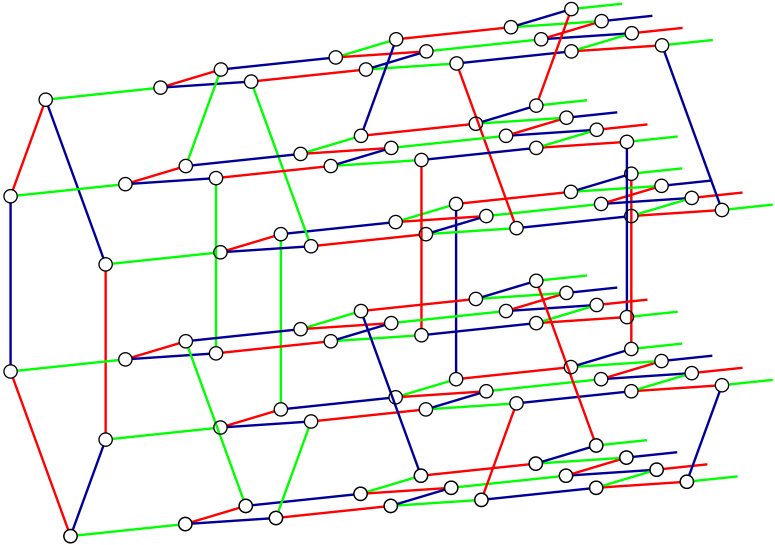
Cayley Graph / Weak Order of the Group of Type D_3



Cayley Graph / Weak Order of the Group of Type H_3



Cayley Graph / Weak Order of the Group of Type $\tilde{A}_2$



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