Loop Identities Obtained by Nuclear Identification

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First Definitions

Definition

Let Q be a quasigroup. An *autotopism* of Q is a triple (α, β, γ) of bijections on Q, satisfying

$$\alpha(y) \cdot \beta(z) = \gamma(yz),$$

for each y, z in Q.

Definition

Let Q be a quasigroup. The permutation $L_x : a \mapsto xa$ is called the *left translation*. The permutation $R_x : a \mapsto ax$ is called the *right translation*.

Definition

A quasigroup Q is called a *loop* if there exists a neutral element in Q (usually denoted by 1).

Moufang Loops Through Autotopisms

Let us consider the identity (so called Moufang identity)

$$xy \cdot zx = x(yz \cdot x).$$

It can be rewritten as

$$L_x(y) \cdot R_x(z) = L_xR_x(yz)$$

Observation

A loop Q satisfies the Moufang identity if and only if the triple (L_x, R_x, L_xR_x) is an autotopism of Q, for each x in Q.

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Nuclei Through Autotopisms

Definition

The *left nucleus* of a loop Q is the set $N_\lambda = \{a \in Q;\ a \cdot xy = ax \cdot y\}$.
The *middle nucleus* of Q is the set $N_\mu = \{a \in Q;\ x \cdot ay = xa \cdot y\}$.
The *right nucleus* of Q is the set $N_\rho = \{a \in Q;\ x \cdot ya = xy \cdot a\}$.

An element a lies in the right nucleus if and only if

$$x \cdot R_a(y) = R_a(xy).$$

Observation

- $a \in N_\rho \Leftrightarrow (\mathrm{id}, R_a, R_a)$ *is an autotopism of* Q.
- $a \in N_\lambda \Leftrightarrow (L_a, \mathrm{id}, L_a)$ *is an autotopism of* Q.
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Nuclear Identification

$$\underbrace{(L_x, \mathrm{id}, L_x)}_{x \in N_\lambda} \cdot \underbrace{(\mathrm{id}, R_x, R_x)}_{x \in N_\rho} = \underbrace{(L_x, R_x, L_x R_x)}_{\text{Moufang}} .$$

Proposition

In a Moufang loop the left and the right nuclei coincide.

$$\underbrace{(L_x, \mathrm{id}, L_x)}_{x \in N_\lambda} \cdot \underbrace{(R_x^{-1}, L_x, \mathrm{id})^{-1}}_{x \in N_\mu} = (L_x R_x, L_x^{-1}, L_x).$$

$(x \cdot yx) \cdot (x \backslash z) = x \cdot yz$ substituting $z \mapsto xz$ gives $(x \cdot yx) \cdot z = x \cdot (y \cdot xz)$

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In a left Bol loop the left and the middle nuclei coincide.

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Nuclear Identities

nuclei	autotopism	law	loop
$\lambda \cdot \mu$	$(L_x R_x^{-1}, L_x, L_x)$	$xy \cdot xz = x(yx \cdot z)$	left extra
$\lambda \cdot \mu^{-1}$	$(L_x R_x, L_x^{-1}, L_x)$	$(x \cdot yx)z = x(y \cdot xz)$	left Bol
$\lambda \cdot \rho$	$(L_x, R_x, L_x R_x)$	$xy \cdot zx = (x \cdot yz)x$	middle Moufang 1
$\lambda \cdot \rho^{-1}$	$(L_x, R_x^{-1}, L_x R_x^{-1})$	$x\backslash(xy \cdot z) = (y \cdot zx)/x$	Buchsteiner
$\lambda^{-1} \cdot \rho$	$(L_x^{-1}, R_x, L_x^{-1} R_x)$	$x(y \cdot zx) = (xy \cdot z)x$	middle extra
$\mu \cdot \lambda$	$(R_x^{-1} L_x, L_x, L_x)$	$((xy)/x)z = x \cdot y(x\backslash z)$	LCC
$\mu^{-1} \cdot \lambda$	$(R_x L_x, L_x^{-1}, L_x)$	$(xy \cdot x)z = x(y \cdot xz)$	left Moufang
$\mu \cdot \rho$	$(R_x^{-1}, L_x R_x, R_x)$	$y(x \cdot zx) = (yx \cdot z)x$	right Moufang
$\mu^{-1} \cdot \rho$	$(R_x, L_x^{-1} R_x, R_x)$	$y(x\backslash(zx)) = (y/x)z \cdot x$	RCC
$\rho \cdot \lambda$	$(L_x, R_x, R_x L_x)$	$xy \cdot zx = x(yz \cdot x)$	middle Moufang 2
$\rho \cdot \mu$	$(R_x^{-1}, R_x L_x, R_x)$	$y(xz \cdot x) = (yx \cdot z)x$	right Bol
$\rho^{-1} \cdot \mu$	$(R_x, R_x L_x^{-1}, R_x)$	$yx \cdot zx = (y \cdot xz)x$	right extra

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I-shifts of Automorphisms

Definition

Denote $I : x \mapsto x\backslash 1$ and $J : x \mapsto 1/x$. Let (α, β, γ) be an autotopism. The triple $(J\gamma I, \alpha, J\beta I)$ is called the *I-shift* of the autotopism.

Observation

The I-shift of $(L_x R_x^{-1}, L_x, L_x)$ is $(JL_x I, L_x R_x^{-1}, JL_x I)$ which looks similar to $(R_x^{-1}, L_x R_x^{-1}, R_x^{-1})$.

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We say that a loop Q has the *weak inverse property* if $JL_x I = R_x^{-1}$.

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More I-shifts

Lemma

In a weak inverse property loop, an I-shift of an autotopism is an autotopism.

Observation

The I-shift of $(R_x, R_xL_x^{-1}, R_x)$ is $(JR_xI, R_x, JR_xL_x^{-1}I)$.
Under the weak inverse property and the assumption $I = J$ we obtain $(L_x^{-1}, R_x, L_x^{-1}R_x)$.

Observation

The I-shift of $(L_x^{-1}, R_x, L_x^{-1}R_x)$ is $(JL_x^{-1}R_xI, L_x^{-1}, JR_xI)$.
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Classes of Equivalence

Proposition

The nuclear identities fall into four classes:

- *left extra, right extra, middle extra*
- *left Moufang, right Bol, middle Moufang 2*
- *left Bol, right Moufang, middle Moufang 1*
- *LCC, RCC, Buchsteiner*

where identities within a class can be obtained one from another using I-shifts, under the WIP and the condition $I = J$.

Lemma

All extra loop identities imply the weak inverse property with $I = J$.

Corollary

In the variety of loops, all the extra laws are equivalent.

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CC and Buchsteiner Loops are Extra

Definition (loop identities)

- *flexibility:* $x \cdot yx = xy \cdot x$,
- *right alternativity:* $x \cdot xy = x^2 y$,
- *right inverse property:* $y/x = y(x\backslash 1)$.

Lemma

For a quasigroup Q, the following are equivalent:

- *Q is extra*
- *Q is flexible LCC*
- *Q is flexible RCC*
- *Q is flexible Buchsteiner*

Proof.

$$(L_x R_x^{-1}, L_x, L_x)(R_x^{-1} L_x, L_x, L_x)^{-1} = (L_x R_x^{-1} L_x^{-1} R_x, \mathrm{id}, \mathrm{id})$$

The last is an autotopism $\Leftrightarrow$ Q is flexible. □

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Further properties

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Let Q be a Buchsteiner loop. Then Q is

$$\text{flexible} \Leftrightarrow \text{left altern.} \Leftrightarrow \text{right altern.} \Leftrightarrow \text{LIP} \Leftrightarrow \text{RIP} \Leftrightarrow \text{extra}$$

Lemma

Let Q be a left Bol loop (or LCC loop). Then Q is

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Moufang loops are exactly left and right Bol loops.

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Extra Loops Are Moufang CC Loops

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Corollary

Extra loops are exactly conjugacy closed Moufang loops.

Definition

Left Bol LCC loop is called *(left) Burn* loop.

Extra Loops Are Moufang CC Loops

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Buchsteiner CC Loops

Proposition

Let Q be a CC loop. Then Q is a Buchsteiner loop if and only if $x^2 \in N(Q)$, for each $x \in Q$.

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A conjugacy closed loop Q with $x^2 \in N(Q)$, for each $x \in Q$, is called *Boolean CC* loop.

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A Buchsteiner loop is LCC if and only if it is RCC.

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