Loop Identities Obtained by Nuclear Identification

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First Definitions

Definition

Let Q be a quasigroup. An *autotopism* of Q is a triple (α, β, γ) of bijections on Q, satisfying

$$\alpha(y) \cdot \beta(z) = \gamma(yz),$$

for each y, z in Q.

Definition

A quasigroup Q is called a *loop* if there exists a neutral element in Q (usually denoted by 1).

Moufang Loops Through Autotopisms

Definition

A loop is called a *Moufang* loop if it satisfies one of the following identities:

$$x(y \cdot xz) = (xy \cdot x)z,$$
$$xy \cdot zx = (x \cdot yz)x,$$
$$xy \cdot zx = x(yz \cdot x),$$
$$y(xz \cdot x) = (yx \cdot z)x.$$

The third identity can be rewritten as

$$L_x(y) \cdot R_x(z) = L_xR_x(yz)$$

Observation

A loop Q is a Moufang loop if and only if the triple (L_x, R_x, L_xR_x) is an autotopism of Q, for each x in Q.

Nuclei Through Autotopisms

Definition

The *left nucleus* of a loop Q is the set $N_\lambda = \{a \in Q;\ a \cdot xy = ax \cdot y\}$.
The *middle nucleus* of Q is the set $N_\mu = \{a \in Q;\ x \cdot ay = xa \cdot y\}$.
The *right nucleus* of Q is the set $N_\rho = \{a \in Q;\ x \cdot ya = xy \cdot a\}$.

An element a lies in the right nucleus if and only if

$$x \cdot R_a(y) = R_a(xy).$$

Observation

- $a \in N_\rho \Leftrightarrow (\mathrm{id}, R_a, R_a)$ *is an autotopism of* Q.
- $a \in N_\lambda \Leftrightarrow (L_a, \mathrm{id}, L_a)$ *is an autotopism of* Q.
- $a \in N_\mu \Leftrightarrow (R_a^{-1}, L_a, \mathrm{id})$ *is an autotopism of* Q.

Nuclear Identification

$$\underbrace{(L_x, \mathrm{id}, L_x)}_{x \in N_\lambda} \cdot \underbrace{(\mathrm{id}, R_x, R_x)}_{x \in N_\rho} = \underbrace{(L_x, R_x, L_x R_x)}_{\text{Moufang}}.$$

Proposition

In a Moufang loop the left and the right nuclei coincide.

$$\underbrace{(L_x, \mathrm{id}, L_x)}_{x \in N_\lambda} \cdot \underbrace{(R_x^{-1}, L_x, \mathrm{id})^{-1}}_{x \in N_\mu} = (L_x R_x, L_x^{-1}, L_x).$$

$(x \cdot yx) \cdot (x \backslash z) = x \cdot yz$ substituting $z \mapsto xz$ gives $(x \cdot yx) \cdot z = x \cdot (y \cdot xz)$

Proposition

In a left Bol loop the left and the middle nuclei coincide.

Loop Laws Induced by the Nuclear Identifications

nuclei	autotopism	law	loop
$\lambda \cdot \mu$	$(L_xR_x^{-1}, L_x, L_x)$	$xy \cdot xz = x(yx \cdot z)$	extra
$\lambda \cdot \mu^{-1}$	(L_xR_x, L_x^{-1}, L_x)	$(x \cdot yx)z = x(y \cdot xz)$	left Bol
$\lambda \cdot \rho$	(L_x, R_x, L_xR_x)	$xy \cdot zx = (x \cdot yz)x$	Moufang
$\lambda \cdot \rho^{-1}$	$(L_x, R_x^{-1}, L_xR_x^{-1})$	$x\backslash(xy \cdot z) = (y \cdot zx)/x$	Buchsteiner
$\lambda^{-1} \cdot \rho$	$(L_x^{-1}, R_x, L_x^{-1}R_x)$	$x(y \cdot zx) = (xy \cdot z)x$	extra
$\mu \cdot \lambda$	$(R_x^{-1}L_x, L_x, L_x)$	$((xy)/x)z = x \cdot y(x\backslash z)$	LCC
$\mu \cdot \lambda^{-1}$	(R_xL_x, L_x^{-1}, L_x)	$(xy \cdot x)z = x(y \cdot xz)$	Moufang
$\mu \cdot \rho$	(R_x^{-1}, L_xR_x, R_x)	$y(x \cdot zx) = (yx \cdot z)x$	Moufang
$\mu^{-1} \cdot \rho$	$(R_x, L_x^{-1}R_x, R_x)$	$y(x\backslash(zx)) = (y/x)z \cdot x$	RCC
$\rho \cdot \lambda$	(L_x, R_x, R_xL_x)	$xy \cdot zx = x(yz \cdot x)$	Moufang
$\rho \cdot \mu$	(R_x^{-1}, R_xL_x, R_x)	$y(xz \cdot x) = (yx \cdot z)x$	right Bol
$\rho^{-1} \cdot \mu$	$(R_x, R_xL_x^{-1}, R_x)$	$xy \cdot xz = x(yx \cdot z)$	extra

WIP Loops

Definition

Denote $I : x \mapsto x\backslash 1$ and $J : x \mapsto 1/x$. We say that a loop Q has the *weak inverse property* if $L_x^{-1} = IR_xJ$.

Definition

Let (α, β, γ) be an autotopism. The triple $(J\gamma I, \alpha, J\beta I)$ is called the I-shift of the autotopism.

Lemma

In a weak inverse property loop, an I-shift of an autotopism is an autotopism.

I-shift Cycles of Autotopisms

Proposition

The loop laws obtained by the nuclear identification come in four cycles:

- $(L_xR_x^{-1}, L_x, L_x)$, $(L_x^{-1}, R_x, L_x^{-1}R_x)$, $(R_x, R_xL_x^{-1}, R_x)$
  — *extra, extra, extra*
- (R_xL_x, L_x^{-1}, L_x), $(R_x, R_xL_x^{-1}, R_x)$, (R_x^{-1}, R_xL_x, R_x)
  — *Moufang, right Bol, Moufang*
- (L_xR_x, L_x^{-1}, L_x), $(R_x, L_x^{-1}R_x, R_x)$, $(L_x, R_x^{-1}, L_xR_x^{-1})$
  — *left Bol, Moufang, Moufang*
- (R_xL_x, L_x^{-1}, L_x), (L_x, R_x, R_xL_x), $(L_x^{-1}, R_x, L_x^{-1}R_x)$
  — *LCC, RCC, Buchsteiner*

Some of these are valid only under the assumption $I = J$.

Equivalence of Laws

Lemma

All extra loop identities imply weak inverse property with $I = J$.

Proposition

In the variety of loops, all the extra laws are equivalent.

Theorem

Let Q be a weak inverse loop. Then the following conditions are equivalent:

- *Q is left conjugacy closed,*
- *Q is right conjugacy closed,*
- *Q is a Buchsteiner loop.*

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