Identités de boucles obtenues par l'identification nucléaire

Loop Identities Obtained by Nuclear Identification

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ČESKÁ ZEMĚDĚLSKÁ UNIVERZITA V PRAZE

Quasigroups

Definition

Let $(G, \cdot)$ be a groupoid. The mapping $L_x : a \mapsto xa$ is called the *left translation* and the mapping $R_x : a \mapsto ax$ the right translation.

Definition (Combinatorial)

A groupoid $(Q, \cdot)$ is called a *quasigroup* if the mappings L_x and R_x are bijections for each $x \in Q$.

Definition (Universal algebraic)

The algebra $(Q, \cdot, /, \backslash)$ is called a *quasigroup* if it satisfies the following identities:

$$x\backslash(x \cdot y) = y \qquad (x \cdot y)/y = x$$
$$x \cdot (x\backslash y) = y \qquad (x/y) \cdot y = x$$

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Isotopisms

Definition

Let $(Q, \cdot)$ and $(R, *)$ be quasigroups. An *isotopism* of Q is a triple (α, β, γ) of bijections from Q onto R, satisfying

$$\alpha(y) * \beta(z) = \gamma(yz),$$

for each y, z in Q.

Example

$\cdot$	0	1	2	3	4
0	1	3	0	2	4
1	4	1	3	0	2
2	2	4	1	3	0
3	0	2	4	1	3
4	3	0	2	4	1

$\alpha = (1\ 3\ 4\ 2)$
$\beta = (1\ 2\ 4\ 3)$
$\gamma = (4\ 3\ 2\ 1\ 0)$

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

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Loops

Definition

A quasigroup Q is called a *loop* if it contains the identity element.

Example (A smallest nonassociative loop)

	1	2	3	4	5
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Center and Nuclei

Definition

Let Q be a loop. We define:

- the *left nucleus* of a loop Q: $N_\lambda = \{a \in Q;\ a \cdot xy = ax \cdot y\}$;
- the *middle nucleus* of Q: $N_\mu = \{a \in Q;\ x \cdot ay = xa \cdot y\}$;
- the *right nucleus* of Q: $N_\rho = \{a \in Q;\ x \cdot ya = xy \cdot a\}$;
- the *nucleus* of Q: $N(Q) = N_\lambda$ if $N_\lambda = N_\mu = N_\rho$.

Definition

The *center* of a loop Q is the set $Z(Q) = \{x \in N_\lambda \cap N_\mu \cap N_\rho;\ xy = yx\ \forall y \in Q\}$.

Theorem

The nucleus and the center are normal subloops.

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Moufang Loops

Definition

A loop Q is called a *Moufang* loop if it satisfies one of the following identities:

$$\begin{aligned} x(y \cdot xz) &= (xy \cdot x)z, \\ xy \cdot zx &= (x \cdot yz)x, \\ xy \cdot zx &= x(yz \cdot x), \\ y(xz \cdot x) &= (yx \cdot z)x. \end{aligned}$$

Examples

- Nonzero octonions
- Octonion units, basis octonions and their inverses
- Nonzero split-octonions
- Paige loops – always simple nonassociative Moufang loops
- Parker's loop

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Smallest Moufang Loop

1	2	3	4	5	6	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$
2	1	4	3	6	5	$\bar{2}$	$\bar{1}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$
3	6	5	2	1	4	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{1}$	$\bar{2}$
4	5	6	1	2	3	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{5}$	$\bar{6}$
5	4	1	6	3	2	$\bar{5}$	$\bar{6}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
6	3	2	5	4	1	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$
$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	1	2	3	4	5	6
$\bar{2}$	$\bar{1}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	2	1	4	3	6	5
$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{1}$	$\bar{2}$	3	6	5	2	1	4
$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{5}$	$\bar{6}$	4	5	6	1	2	3
$\bar{5}$	$\bar{6}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	5	4	1	6	3	2
$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	6	3	2	5	4	1

Properties of Moufang Loops

Moufang loops are

- flexible $x(yx) = (xy)x$
- left alternative $x(xy) = (xx)y$
- right alternative $y(xx) = (yx)x$
- diassociative (every two-generated subloop is associative)

They have

- two-sided inverses: $x^{-1} = 1/x = x\backslash 1$
- left inverse property: $x^{-1}(xy) = y$
- right inverse property: $(yx)x^{-1} = y$
- antiautomorphic inverse property: $(xy)^{-1} = y^{-1}x^{-1}$
- Lagrange property (2003)

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Bol loops

Definition

A loop Q is called a *(left) Bol* loop, if it satisfies $x(y \cdot xz) = (x \cdot yx)z$.

Example

Let A and B be positively definite Hermitian matrices. There exist unique P, a positive definite Hermitian matrix and U, a unitary matrix, such that $AB = PU$. The set of positively definite Hermite matrices with $A * B = P$ is a Bol loop.

Example

Einstein's velocity addition formula

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Extra Loops

Definition

A loop is said to be *extra* if it satisfies one of these identities:

- $xy \cdot xz = x(yx \cdot z)$,
- $x(y \cdot zx) = (xy \cdot z)x$,
- $yx \cdot zx = (y \cdot xz)x$.

Examples

- 16-element loop of octonion generators and their inverses
- 16-element loop of split-octonion generators and their inverses

Theorem

Extra loops are Moufang loops with $x^2 \in N(Q)$ *for each* x *in* Q.

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Conjugacy Closed Loops

Definition

A loop Q is *left conjugacy closed* if, for each x, y in Q there exists a in Q such that

$$L_x L_y L_x^{-1} = L_a .$$

Observation

Necessarily $a = xy/x$ and LCC is defined by $x(y \cdot xz) = (xy/x) \cdot z$.

Example

1	2	3	$\bar{1}$	$\bar{2}$	$\bar{3}$
2	3	1	$\bar{3}$	$\bar{1}$	$\bar{2}$
3	1	2	$\bar{2}$	$\bar{3}$	$\bar{1}$
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Moufang Loops Through Autotopisms

Let us consider the identity

$$xy \cdot zx = x(yz \cdot x).$$

It can be rewritten as

$$L_x(y) \cdot R_x(z) = L_xR_x(yz)$$

Observation

A loop Q satisfies the Moufang identity if and only if the triple (L_x, R_x, L_xR_x) is an autotopism of Q, for each x in Q.

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Nuclei Through Autotopisms

The right nucleus of Q is the set $N_\rho = \{a \in Q;\ x \cdot ya = xy \cdot a\}$.
An element a lies in the right nucleus if and only if

$$x \cdot R_a(y) = R_a(xy).$$

Observation

- $a \in N_\rho \Leftrightarrow (\mathrm{id}, R_a, R_a)$ *is an autotopism of* Q.
- $a \in N_\lambda \Leftrightarrow (L_a, \mathrm{id}, L_a)$ *is an autotopism of* Q.
- $a \in N_\mu \Leftrightarrow (R_a^{-1}, L_a, \mathrm{id})$ *is an autotopism of* Q.

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Nuclear Identification

$$\underbrace{(L_x, \mathrm{id}, L_x)}_{x \in N_\lambda} \cdot \underbrace{(\mathrm{id}, R_x, R_x)}_{x \in N_\rho} = \underbrace{(L_x, R_x, L_x R_x)}_{\text{Moufang}}.$$

Proposition

In a Moufang loop the left and the right nuclei coincide.

$$\underbrace{(L_x, \mathrm{id}, L_x)}_{x \in N_\lambda} \cdot \underbrace{(R_x^{-1}, L_x, \mathrm{id})^{-1}}_{x \in N_\mu} = (L_x R_x, L_x^{-1}, L_x).$$

$(x \cdot yx) \cdot (x \backslash z) = x \cdot yz$ substituting $z \mapsto xz$ gives $(x \cdot yx) \cdot z = x \cdot (y \cdot xz)$

Proposition

In a left Bol loop the left and the middle nuclei coincide.

Nuclear Identification

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Nuclear Identities

nuclei	autotopism	law	loop
$\lambda \cdot \mu$	$(L_x R_x^{-1}, L_x, L_x)$	$xy \cdot xz = x(yx \cdot z)$	left extra
$\lambda \cdot \mu^{-1}$	$(L_x R_x, L_x^{-1}, L_x)$	$(x \cdot yx)z = x(y \cdot xz)$	left Bol
$\lambda \cdot \rho$	$(L_x, R_x, L_x R_x)$	$xy \cdot zx = (x \cdot yz)x$	middle Moufang 1
$\lambda \cdot \rho^{-1}$	$(L_x, R_x^{-1}, L_x R_x^{-1})$	$x\backslash(xy \cdot z) = (y \cdot zx)/x$	Buchsteiner
$\lambda^{-1} \cdot \rho$	$(L_x^{-1}, R_x, L_x^{-1} R_x)$	$x(y \cdot zx) = (xy \cdot z)x$	middle extra
$\mu \cdot \lambda$	$(R_x^{-1} L_x, L_x, L_x)$	$((xy)/x)z = x \cdot y(x\backslash z)$	LCC
$\mu^{-1} \cdot \lambda$	$(R_x L_x, L_x^{-1}, L_x)$	$(xy \cdot x)z = x(y \cdot xz)$	left Moufang
$\mu \cdot \rho$	$(R_x^{-1}, L_x R_x, R_x)$	$y(x \cdot zx) = (yx \cdot z)x$	right Moufang
$\mu^{-1} \cdot \rho$	$(R_x, L_x^{-1} R_x, R_x)$	$y(x\backslash(zx)) = (y/x)z \cdot x$	RCC
$\rho \cdot \lambda$	$(L_x, R_x, R_x L_x)$	$xy \cdot zx = x(yz \cdot x)$	middle Moufang 2
$\rho \cdot \mu$	$(R_x^{-1}, R_x L_x, R_x)$	$y(xz \cdot x) = (yx \cdot z)x$	right Bol
$\rho^{-1} \cdot \mu$	$(R_x, R_x L_x^{-1}, R_x)$	$yx \cdot zx = (y \cdot xz)x$	right extra

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I-shifts of Automorphisms

Definition

Denote $I : x \mapsto x\backslash 1$ and $J : x \mapsto 1/x$. Let (α, β, γ) be an autotopism. The triple $(J\gamma I, \alpha, J\beta I)$ is called the *I-shift* of the autotopism.

Observation

The I-shift of $(L_x R_x^{-1}, L_x, L_x)$ is $(JL_x I, L_x R_x^{-1}, JL_x I)$ which looks similar to $(R_x^{-1}, L_x R_x^{-1}, R_x^{-1})$.

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More I-shifts

Lemma

In a weak inverse property loop, an I-shift of an autotopism is an autotopism.

Observation

The I-shift of $(R_x, R_xL_x^{-1}, R_x)$ is $(JR_xI, R_x, JR_xL_x^{-1}I)$.
Under the weak inverse property and the assumption $I = J$ we obtain $(L_x^{-1}, R_x, L_x^{-1}R_x)$.

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The I-shift of $(L_x^{-1}, R_x, L_x^{-1}R_x)$ is $(JL_x^{-1}R_xI, L_x^{-1}, JR_xI)$.
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Classes of Equivalence

Proposition

The nuclear identities fall into four classes:

- *left extra, right extra, middle extra*
- *left Moufang, right Bol, middle Moufang 2*
- *left Bol, right Moufang, middle Moufang 1*
- *LCC, RCC, Buchsteiner*

where identities within a class can be obtained one from another using I-shifts, under the WIP and the condition $I = J$.

Lemma

All extra loop identities imply the weak inverse property with $I = J$.

Corollary

In the variety of loops, all the extra laws are equivalent.

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CC and Buchsteiner Loops are Extra

Definition (loop identities)

- *flexibility:* $x \cdot yx = xy \cdot x$,
- *right alternativity:* $x \cdot xy = x^2 y$,
- *right inverse property:* $y/x = y(x \backslash 1)$.

Lemma

For a quasigroup Q, the following are equivalent:

- *Q is extra*
- *Q is flexible LCC*
- *Q is flexible RCC*
- *Q is flexible Buchsteiner*

Proof.

$$(L_x R_x^{-1}, L_x, L_x)(R_x^{-1} L_x, L_x, L_x)^{-1} = (L_x R_x^{-1} L_x^{-1} R_x, \mathrm{id}, \mathrm{id})$$

The last is an autotopism $\Leftrightarrow$ Q is flexible. □

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Further properties

Lemma

Let Q be a Buchsteiner loop. Then Q is

flexible ⇔ left altern. ⇔ right altern. ⇔ LIP ⇔ RIP ⇔ extra

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Let Q be a left Bol loop (or LCC loop). Then Q is

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Moufang loops are exactly left and right Bol loops.

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Extra Loops Are Moufang CC Loops

Lemma

Extra loops are Moufang loops

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Corollary

Extra loops are exactly conjugacy closed Moufang loops.

Definition

Left Bol LCC loop is called *(left) Burn* loop.

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Buchsteiner CC Loops

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Let Q be a CC loop. Then Q is a Buchsteiner loop if and only if $x^2 \in N(Q)$, for each $x \in Q$.

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A conjugacy closed loop Q with $x^2 \in N(Q)$, for each $x \in Q$, is called *Boolean CC* loop.

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Intersection Semilattice of Nuclear Varieties