Geometry Monoids and Geometry Groups

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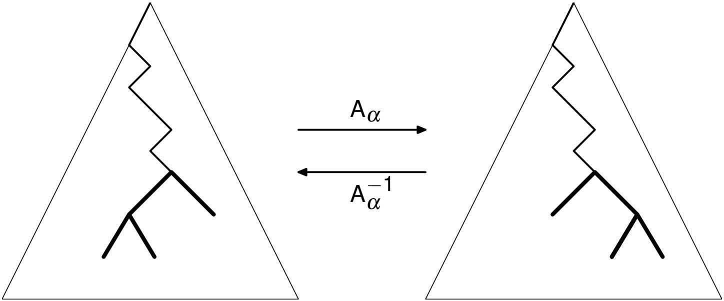
Geometry operators

Definitions

An *address* is a word on $\{0,1\}$ that encodes a position of a subterm in a term (0 stands for left and 1 stands for right).
If α is an address than the operator A_α is a partial mapping that sends a term with a subterm of form $(t_1 \cdot t_2) \cdot t_3$ at the address α to the term with $t_1 \cdot (t_2 \cdot t_3)$ at the address α if such a subterm exists.



Geometry Monoids and Geometry groups

Definitions

The *geometry monoid* of the associativity is the monoid generated by the partial mappings A_α and A_α^{-1}, for all $\alpha \in \{0,1\}^*$.
The *geometry group* of the associativity is the geometry monoid of the associativity quotioned by the relations $\mathsf{A}_\alpha \circ \mathsf{A}_\alpha^{-1} = \mathsf{A}_\alpha^{-1} \circ \mathsf{A}_\alpha = \mathrm{id}$, for all $\alpha \in \{0,1\}^*$.

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The *positive geometry monoid* of the associativity is the monoid generated by the partial mappings A_α, for all $\alpha \in \{0,1\}^*$.

Geometry Monoids and Geometry groups

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The *geometry monoid* of the associativity is the monoid generated by the partial mappings $\mathtt{A}_\alpha$ and $\mathtt{A}_\alpha^{-1}$, for all $\alpha \in \{0,1\}^*$.
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Definition

The *positive geometry monoid* of the associativity is the monoid generated by the partial mappings $\mathtt{A}_\alpha$, for all $\alpha \in \{0,1\}^*$.

Relations in the positive geometry monoid

Fact

For all addresses α, β, γ, one has

$$\mathsf{A}_{\alpha 1\gamma} \circ \mathsf{A}_{\alpha 0\beta} = \mathsf{A}_{\alpha 0\beta} \circ \mathsf{A}_{\alpha 1\gamma}$$

$$\mathsf{A}_{\alpha} \circ \mathsf{A}_{\alpha 00\beta} = \mathsf{A}_{\alpha 0\beta} \circ \mathsf{A}_{\alpha}$$

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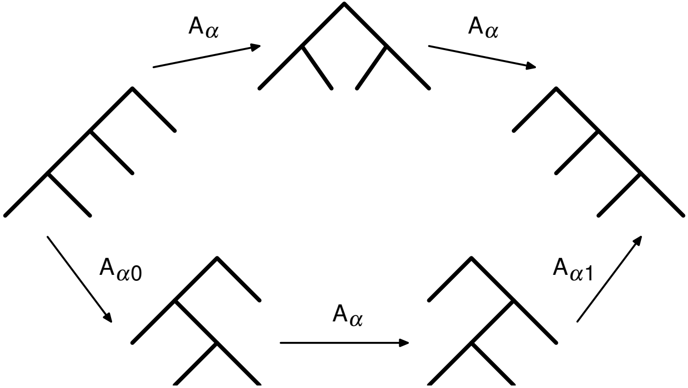
MacLane's penthagon



Syntactical Monoid

Definition

The *syntactical monoid* of the associativity is the monoid with the presentation

$$\begin{aligned}\Big\langle \{0,1\}^*;\ \alpha 1\gamma \cdot \alpha 0\beta &= \alpha 0\beta \cdot \alpha 1\gamma \\ \alpha \cdot \alpha 00\beta &= \alpha 0\beta \cdot \alpha \\ \alpha \cdot \alpha 01\beta &= \alpha 11\beta \cdot \alpha \\ \alpha \cdot \alpha &= \alpha 1 \cdot \alpha \cdot \alpha 0 \Big\rangle\end{aligned}$$

Properties of the Syntactical Monoid

Theorem

The syntactical monoid of the associativity has the following properties:

- *it is left and right cancellative;*
- *the left divisibility forms a lattice;*
- *the right divisibility forms a lattice;*
- *it embeds into its group of fractions;*
- *its word problem is solvable (and hence also the word problem of the group of fractions);*
- *it is isomorphic to the positive geometry monoid of the associativity.*

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Examples of known geometry groups and monoids

Examples

- Associativity – Thompson group F
- Associativity and commutativity – Thompson group V
- Left distributivity $x \cdot yz = xy \cdot xz$ (P. Dehornoy)
- Central doubling $x \cdot yz = xy \cdot yz$ (P. Dehornoy)
- Left distributivity and left idempotency $xy = xx \cdot y$ (P. Jedlička)

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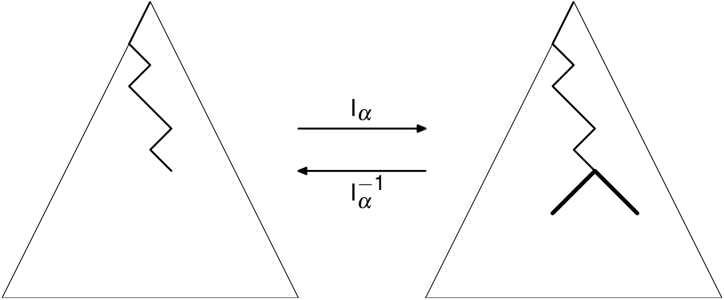
Idempotent operators



$$I_\alpha \circ I_\alpha = I_{\alpha 0} \circ I_{\alpha 1} \circ I_\alpha = I_{\alpha 1} \circ I_{\alpha 0} \circ I_\alpha$$

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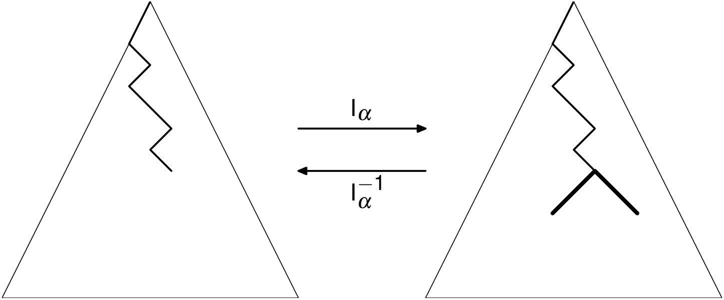

$$I_\alpha \circ I_\alpha = I_{\alpha 0} \circ I_{\alpha 1} \circ I_\alpha = I_{\alpha 1} \circ I_{\alpha 0} \circ I_\alpha$$

Presentation of the Syntactical Monoid of LDLI

$$\begin{aligned}
\mathsf{LD}_{\gamma 0\alpha}\cdot\mathsf{LD}_{\gamma 1\beta} &= \mathsf{LD}_{\gamma 1\beta}\cdot\mathsf{LD}_{\gamma 0\alpha}\\
\mathsf{I}_{\gamma 0\alpha}\cdot\mathsf{LD}_{\gamma 1\beta} &= \mathsf{LD}_{\gamma 1\beta}\cdot\mathsf{I}_{\gamma 0\alpha}\\
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\end{aligned}$$

$$\begin{aligned}
\mathsf{LD}_{\gamma}\cdot\mathsf{LD}_{\gamma 1}\cdot\mathsf{LD}_{\gamma} &= \mathsf{LD}_{\gamma 1}\cdot\mathsf{LD}_{\gamma}\cdot\mathsf{LD}_{\gamma 1}\cdot\mathsf{LD}_{\gamma 0}\\
\mathsf{I}_{\gamma\alpha}\cdot\mathsf{I}_{\gamma} &= \mathsf{I}_{\gamma}\cdot\mathsf{I}_{\gamma 0\alpha}\cdot\mathsf{I}_{\gamma 1\alpha}\\
\mathsf{LD}_{\gamma\alpha}\cdot\mathsf{I}_{\gamma} &= \mathsf{I}_{\gamma}\cdot\mathsf{LD}_{\gamma 0\alpha}\cdot\mathsf{LD}_{\gamma 1\alpha}\\
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\mathsf{LD}_{\gamma}\cdot\mathsf{I}_{\gamma 0} &= \mathsf{I}_{\gamma 10}\cdot\mathsf{LD}_{\gamma}\cdot\mathsf{LD}_{\gamma 0}\\
\mathsf{I}_{\gamma 11\alpha}\cdot\mathsf{LD}_{\gamma} &= \mathsf{LD}_{\gamma}\cdot\mathsf{I}_{\gamma 11\alpha}
\end{aligned}$$

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