ON THE VARIETY OF LEFT DIVISIBLE LEFT DISTRIBUTIVE GROUPOIDS

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ABSTRACT. We investigate the following classes of the left distributive groupoids: left distributive left quasigroups, left divisible left distributive groupoids and left cancellative left idempotent left distributive groupoids and we conjecture that they are described by the same equational theories. More precisely, we translate the problem into the existence problem of a given groupoid.

Many groupoids that satisfy the left distributivity

$$x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z) \tag{LD}$$

satisfy the idempotency

$$x \cdot x = x \tag{I}$$

too. An example of such a left distributive idempotent (LDI) groupoid is a group G with the conjugacy, i.e. the operation $x\hat{\ }y = yxy^{-1}$. It was an open question for a long time whether groupoids of group conjugacy (GC) generate all the variety LDI or if there exists an equation that holds in GC and not in LDI. This question was solved independently by D. Larue [7] and A. Drápal, T. Kepka and R. Musílek [3]. Moreover, Larue showed the following characterisation:

Theorem. *(Larue) The following varieties coincide:*

- *the variety generated by GC;*
- *the variety generated by the left cancellative LDI groupoids;*
- *the variety generated by the left divisible LDI groupoids;*
- *the variety generated by the LDI left quasigroups (i.e. left cancellative left divisible LDI groupoids).*

Does an analogous characterisation hold without the idempotecy? It is easy to see (cf. Corollary 1.2) that all left divisible left distributive (LDLD) groupoids satisfy the following identity:

$$(x \cdot x) \cdot y = x \cdot y \tag{LI}$$

called the left idempotency. It would be therefore tempting to replace the idempotency in Larue's theorem by the left idempotency. Indeed, T. Kepka and P. Němec proved in [6] that every left cancellative left distributive left idempotent (LCLDLI) groupoid embeds into a left distributive left quasigroup (LDLQ). Moreover, P. Dehornoy showed in [2] that the variety of LDLQ is generated by groups with the half-conjugacy, that means an operation $(a, x)\hat{\ }(b, y) = (axa^{-1}b, y)$, where $a, b \in G$ and $x, y \in X \subseteq G$.

In order to have a complete analogy of Larue's theorem, it remains to prove that the variety LDLD is the same as LDLQ=LCLDLI. And this is the aim of the

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article. Although we do not prove the intended inclusion, we give some evidence and we show an easy to understand sufficient condition for this to hold.

1. Some evidence

In the begining we study the histrocally first discovered identity which holds in GC and not in LDI. It is easy to show that it holds in LCLDLI too and we show that it holds in LDLD as well, giving an evidence to the inclusion LDLD$\subseteq$ LCLDLI.

We work with non-associative algebras and therefore many parentheses are formally needed. Nevertheless, when working with LD groupoids, it is common to spare them. We write $xy \cdot z$ instead of $(x \cdot y) \cdot z$ and omitted parentheses mean branching to the right, i.e. $xyz = x \cdot yz$.

Lemma 1.1. *Let G be an LDLD groupoid. Let us denote by $a\backslash b$ an element satisfying $a \cdot a\backslash b = b$. Then, for each $a, b, c \in G$,*

$$ab \cdot c = a \cdot b \cdot a\backslash c \tag{1}$$

Proof. $ab \cdot c = ab \cdot (a \cdot a\backslash c) = a \cdot b \cdot a\backslash c$. □

Corollary 1.2. *Every left divisible left distributive groupoid is left idempotent.*

Proof. $xx \cdot y = x \cdot (x \cdot x\backslash y) = xy$ □

The shortest pair of terms, that are equivalent in GC and not in LDI, is $(ab \cdot b)ac$ and $(ab) \cdot (ba \cdot c)$. Larue [7] proved that they are equal in LCLDLI (we denote it by $\overset{\text{LCLDLI}}{=}$); actually, the proof does not mention the left idempotency; nevertheless, every idempotency used in the proof is a left one. And now it turns out that they are equivalent in LDLD too.

Proposition 1.3. *Let G be an LDLD groupoid. Then, for every $a, b, c \in G$,*

$$(ab \cdot b)ac = (ab) \cdot (ba \cdot c) \tag{2}$$

Proof.
$$\begin{aligned}
(ab \cdot b)ac &= (a \cdot b \cdot a\backslash b)ac && \text{(Lemma 1.1)}\\
&= a(b \cdot a\backslash b)c && \text{(LD)}\\
&= a \cdot b \cdot a\backslash b \cdot b\backslash c && \text{(Lemma 1.1)}\\
&= (ab) \cdot a \cdot a\backslash b \cdot b\backslash c && \text{(LD)}\\
&= (ab) \cdot (a \cdot a\backslash b) \cdot a \cdot b\backslash c && \text{(LD)}\\
&= (ab) \cdot b \cdot a \cdot b\backslash c && \text{definition}\\
&= (ab) \cdot (ba \cdot c) && \text{(Lemma 1.1)}
\end{aligned}$$
□

Larue [7] found an infinite family of identities that hold in the group conjugacy and not in LDI. And they hold in LDLD as well; in fact, the proof in [7] did not use the idempotency, just the distributivity and $(ab \cdot b)ac = (ab) \cdot (ba \cdot c)$ and therefore it remains valid in the non-idempotent case too.

2. Equality of varieties

In this section we try to prove that the variety generated by the LCLDLI coincides with the variety generated by the LDLD. As we already mentioned in the introduction, one inclusion is already known and we would like to prove that, whenever an identity holds in every LCLDLI groupoid then it holds in every LDLD groupoid too.

First we measure how far is an LD groupoid from being left cancellative.

Lemma 2.1 (Kepka)**.** [5] *Let G be an LD groupoid. The relation $\sim$ defined by $a \sim b \Leftrightarrow x_1x_2 \cdots x_na = x_1x_2 \cdots x_nb$, for some $x_1, \ldots, x_n$ in G, is the smallest congruence on G such that $G/_\sim$ is left cancellative.*

We will try to follow the same argumentation as Larue used when proving the similar theorem for LDI; we just replace every occurence of the idempotency in his proof by the left idempotency. However, there is a small glitch that we are not able to overcome in the moment and we need an additional condition.

Conjecture 2.2. *Let G be an LDLD groupoid. Then the mapping $a \mapsto a^2$ is onto.*

What does it mean? In every left divisible groupoid the equation $ax = a$ has a solution. The conjecture states how does a solution look like. If we denote by $\sqrt{a}$ an inverse image of the squaring mapping then

$$a = (\sqrt{a})^2 = (\sqrt{a})^2 \cdot \sqrt{a} = a \cdot \sqrt{a}.$$

It turns out that the conjecture is necessary for Larue's proof to hold in the case of LDLI. Of course, in the case of idempotent groupoids, squaring is trivially onto.

Lemma 2.3. *Suppose that Conjecture 2.2 holds. For any generators $g_1, \dots, g_m$ from X_n and words u, v from T_n, the equality $g_m \cdots g_1 u \stackrel{\text{LDLI}}{=} g_m \cdots g_1 v$ implies $u \stackrel{\text{LDLD}}{=} v$.*

Proof. Suppose that we have $g_m \cdots g_1 u \stackrel{\text{LDLI}}{=} g_m \cdots g_1 v$ for some generators $g_1, \dots, g_m$ from X_n and words u, v in T_n. Let G be an LDLD groupoid. The words u and v are words in variables $x_1, \dots, x_n$, which can be written as $u(x_1, \dots, x_n)$, respectively $v(x_1, \dots, x_n)$. Let us take $a_1, \dots, a_n$ in G arbitrary. We want to show $u(a_1, \dots, a_n) = v(a_1, \dots, a_n)$.

Each g_i belongs to X_n and hence we can write it as some x_j. Denote $g_i = x_{\sigma(i)}$. We claim by induction that, for each $0 \le i \le m$, there exist $b_1, \dots, b_i$ from G such that

$$\begin{aligned} u(a_1, \dots, a_n) &= b_{\sigma(i)} b_{\sigma(i-1)} \cdots b_{\sigma(1)} u(b_1, \dots, b_n), \\ v(a_1, \dots, a_n) &= b_{\sigma(i)} b_{\sigma(i-1)} \cdots b_{\sigma(1)} v(b_1, \dots, b_n). \end{aligned}$$

The result is vaccuously true for $i = 0$. Suppose now that all such b_k exist for some i and let us prove the result for $i + 1$. For each $1 \le k \le n$ we put b'_k to be an element satisfying $b_{\sigma(i+1)} b'_k = b_k$, such elements exist due to the left divisibility. Moreover, we want ${b'_{\sigma(i+1)}}^2 = b_{\sigma(i+1)}$, which is guaranteed by Conjecture 2.2. Now

$$\begin{aligned} u(a_1, a_2, \dots, a_n) &= b_{\sigma(i)} b_{\sigma(i-1)} \cdots b_{\sigma(1)} u(b_1, \dots, b_n) \\ &= b_{\sigma(i+1)} b'_{\sigma(i)} \cdot b_{\sigma(i+1)} b'_{\sigma(i-1)} \cdots b_{\sigma(i+1)} b'_{\sigma(1)} \cdot u(b_1, \dots, b_n) \\ &= b_{\sigma(i+1)} \cdot b'_{\sigma(i)} b'_{\sigma(i-1)} \cdots b'_{\sigma(1)} \cdot u(b_1, \dots, b_n) \\ &= {b'_{\sigma(i+1)}}^2 \cdot b'_{\sigma(i)} b'_{\sigma(i-1)} \cdots b'_{\sigma(1)} \cdot u(b_1, \dots, b_n) \\ &= b'_{\sigma(i+1)} b'_{\sigma(i)} b'_{\sigma(i-1)} \cdots b'_{\sigma(1)} \cdot u(b_1, \dots, b_n) \end{aligned}$$

and similarly for v, which finishes the induction.

Now,

$$\begin{aligned} u(a_1, \dots, a_n) &= b_{\sigma(m)} b_{\sigma(m-1)} \cdots b_{\sigma(1)} \cdot u(b_1, \dots, b_n) \\ &= (x_{\sigma(m)} x_{\sigma(m-1)} \cdots x_{\sigma(1)} \cdot u)(b_1, \dots, b_n) = g_m g_{m-1} \cdots g_1 u(b_1, \dots, b_n) \end{aligned}$$

and similarly for v. Since $g_m g_{m-1} \cdots g_1 u \stackrel{\text{LDLI}}{=} g_m g_{m-1} \cdots g_1 v$, we get $u(a_1, \dots, a_n) = v(a_1, \dots, a_n)$ as desired. □

Proposition 2.4 (D. Larue)**.** [7] *For any words $w_1, \dots, w_k \in T_n$ there exist integers m, l, generators $g_1, \dots, g_m$ and terms $p_1, \dots, p_l$ such that*

$$g_m \cdots g_2 g_1 u \stackrel{\text{LD}}{=} p_l \cdots p_1 w_k \cdots w_1 u$$

for any $u \in T_n$.

Proposition 2.5. *If conjecture 2.2 holds then, for any terms $u, v \in T_n$, $u \overset{\mathrm{LCLDLI}}{=} v$ implies $u \overset{\mathrm{LDLD}}{=} v$.*

Proof. Suppose $u \overset{\mathrm{LCLDLI}}{=} v$. According to Lemma 2.1, there exist terms $w_1, \dots, w_k$ such that $w_k \cdots w_1 u \overset{\mathrm{LDLI}}{=} w_k \cdots w_1 v$.

According to Proposition 2.4, there exist generators $g_1, \dots, g_m$ and terms $p_1, \dots, p_l$ such that $g_m \cdots g_1 \overset{\mathrm{LD}}{=} p_l \cdots p_1 w_k \cdots w_1 z$ for all z. Now

$$g_m \cdots g_1 u \overset{\mathrm{LD}}{=} p_l \cdots p_1 w_k \cdots w_1 u \overset{\mathrm{LDLI}}{=} p_l \cdots p_1 w_k \cdots w_1 v \overset{\mathrm{LD}}{=} g_m \cdots g_1 v$$

and we apply Lemma 2.3. □

3. Coda

The crucial point of the article is Conjecture 2.2; nothing guarantees that it is true, perhaps there exists a counterexample. To better find some, we express the conjecture in a different way, using the structure of LDLI groupoids. The most important structural rôle is played by an equivalence called ip_G.

Proposition 3.1. [4] *Let G be an LDLI groupoid. We define ip_G to be the smallest equivalence on G containing the pairs (a, a^2). Then*

- *For all a, b, c in G, if $(a, b) \in \mathrm{ip}_G$ then $ac = bc$.*
- *ip_G is a congruence of G with its classes being subgroupoids of G.*

Proposition 3.2. *Let G be an LDLI groupoid. The following conditions are equivalent:*

(i) *The mapping $a \mapsto a^2$ is onto.*
(ii) *For each a in G, there exists an element x in G, satisfying $a \cdot x = a$ and $(a, x) \in \mathrm{ip}_G$.*
(iii) *Every class of ip_G is a left divisible groupoid.*

Proof. (i)⇒(ii): Already proven below Conjecture 2.2.
(ii)⇒(i): Let x be an element satifying $ax = a$ and $(a, x) \in \mathrm{ip}_G$. According to Proposition 3.1, we have $xx = ax$.
(ii)⇒(iii): Let b and c be ip_G-equivalent elements in G. We want to find an element x within the same congruence class, satisfying $bx = c$. But there exists x, satistying $cx = c$ and $(c, x) \in \mathrm{ip}_G$. And, according to Proposition 3.1, we have $cx = bx$.
(iii)⇒(ii): Evident. □

This narrows the possibilities where to look for a possible counterexample for Conjecture 2.2; a popular means how to find a counterexample is to look for one using an automated model builder. This is impossible here since model builders can handle finite objects only.

Corollary 3.3. *Every finite LDLD groupoid satisfies Conjecture 2.2.*

Proof. Let G be a finite left divisible LDLI groupoid. Finite left divisible groupoids are left cancellative. Left cancellativity carries to subgroupoids. Hence every class of ip_G is finite left cancellative and thus left divisible. Therefore, according to Proposition 3.2, the groupoid G satisfies Conjecture 2.2. □

How to disprove the conjecture, if it were wrong? It would be, perhaps, necessary to find some identities that hold in LCLDLI and not in LDLD. There is, actually, a new family of identities, that hold in GC and do not hold in LDI: they were discovered by J. Barboriková [1]. So far, we do not know, whether they bring anything to our study of LDLI groupoids.

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