CHARLES UNIVERSITY, PRAGUE
FACULTY OF MATHEMATICS AND PHYSICS

Divisibility lattices in positive braid monoids

Přemysl Jedlička

Summary of Ph.D. Thesis

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UNIVERZITA KARLOVA, PRAHA
MATEMATICKO-FYZIKÁLNÍ FAKULTA

Svazy dělitelnosti v monoidech kladných pletenců

Přemysl Jedlička

Autoreferát doktorské dizertační práce

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Výsledky tvořící dizertační práci byly získány během studia na MFF UK v Praze a na Univerzitě v Caen v letech 2001–2004.

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Introduction

This text comprises two parts, namely the study of the semidirect products of lattices with applications in the Coxeter groups and in the Garside monoids, and the one of the free self-distributif idempotent groupoids (free LDI groupoids), which has as its elements the equivalence classes structured as lattices. The common point that connects this enough independent parts is the notion of lattice and of the confluence and, more generally, the type of combinatorial and algebraic arumentation used.

This first part is centred on the notion of semidirect product of lattices, whis is the analog, in the world of lattices, of the semidirect product of groups. As in the case of groups, there exists an internal version and an external version of the semidirect product of lattices, and this one is constructed using an action of one of the lattices on the other, the direct product corresponds to the case where the action is trivial. Since a lattice possesses two basic operations (meet and join), the construction of the semidirect product requiers *a priori* two applications from the first lattice into the endomorphisms of the second one. Actually, one mapping is sufficient for determine the order of the product and therefore for determine the second mapping.

Chapter 1 of the thesis describes the construction of the semidirect product of lattices, as well as the one of the semidirect product of semilattices (the case where only one lattice operation is defined). Many examples are analysed in details. The main result from the point of view of further developments is Proposition 1.15 which shows that, for determine a semidirect product of discrete lattices (and particulary of finite lattice), there suffices to know how to enumerate a set of triples called special, which code in a certain sense the immediate successor relation in the product. Another result is a characterisation of the smallest class of lattices closed under semidirect products:

Proposition 1.25 *Let L be a finite lattice. Then the following conditions are equivalent:*
(i) *the lattice L belongs to the smallest class that contains the two element lattice and that is closed under sublattices, semidirect products and under isomorphic images;*
(ii) *the lattice L does not contain any sublattice that maps onto a simple lattice;*
(iii) *the lattice L belongs to the smallest class that contains the two element latice and that is closed under sublattices, under short exact sequences and under isomorphic images;*
(iv) *the lattice L belongs to the greatest class of lattices closed under sublattices and under homomorphic images that does not contain any simple lattice.*

Chapter 2 describes an application of the semidirect product of lattices in the Coxeter groups. This application is in fact the starting point of all twe work and the main motivation for which the general semidirect product was introduced. The Coxeter groups are a class of groups containing the symmetric groups and, more genrally, the reflection groups of an affine space, and they have been object of multiple works [27], [4] or [5]. Each Coxeter group W is equiped with a partial order relation called *the weak order* $\preccurlyeq$ [3], which gives a semilattice structure on W in all cases, and a lattice structure in the case when W is finite. Our goal is to describe an explicit construction of this (semi)lattice structure, and this is where the semidirect product intervene. Among all the subgroups of a Coxeter group, there exists subgroups called *parabolic* which are themselves Coxeter groups, and which are indexed by the subsets of the set of the canonical generators of W. If W_J is a parabolic subgroup of W, there exists a natural decomposition of the elements of W associated with W_J, and it shows that this decomposition gives a congruence of the weak order (semi)lattice of W. Moreover, the classes of this congruence are mutually isomorphic and they have themselves a (semi)-lattice structure. We show the following general result:

Theorem 2.10 *Let (W, S) be a Coxeter system and let J be a subset of S. We denote by W_J the subgroup generated by J and by W^J a left coset associated with W_J. Then the semilattice $(W, \preccurlyeq)$ is isomorphic to a semidirect product of semilattices $(W_J, \preccurlyeq)$ and $(W^J, \preccurlyeq)$.*

Moreover, the preceding result is effectif in the measure that we can, in a great number of examples, completely describe the considered semidirect products exhaustively enumerating the special triples. We treat also the case of the types A_n, B_n, D_n, I_m, as well as the one of $\widetilde{A}_2$: for every infite family, the weak ordre lattice is constructed inductively starting with the one of the preceding group of the family.

Other results of decompositions exist in the litterature [41], [23]. The specificity of our result is to enable an inductive construction (to construct the big lattice starting with the small lattices, which are the only known) and not only a decomposition (to exprime the elements of the big lattice, supposed to be known, in terms of the small lattices). Moreover, it seems that only the case of finite Coxeter groups has been considered, although our approach applicates to the infinite case as well.

Chapter 3 is sacrificed to the study ot the divisibility lattices in the Garside monoids. To each Coxeter group we associate so called Artin group, or Artin-Tits group, the presentation of which is obtained from the standard presentation of the Coxeter groups removing the torsion relations $s^2 = 1$, and a monoid of the same presentation. If W is a finite Coxeter group, and if M is the corresponding Artin monoid, there exists by construction a surjective homomorphism from M on W, and there exists also a canonical set section of this homomorphism. The image under this section of the maximal element w_0 of W is an element Δ of M which has important properties, and particulary the divisors of Δ in the sense of monoid M form a lattice isomorphic to the weak order lattice on W. We study here the lattices that appear as the lattices of the divisors of the elements Δ^k as well as the divisibility lattice of the monoid M :

Proposition 3.11 *Let M be an irreducible Artin monoid of spherical type having at least two atoms. Then the divisibility lattice of M is simple.*

The notion of Artin monoid is generalised to the Garside monoid defined as the monoid having an element called *Garside element,* divisors of which form a lattice [17]. We describe examples of Garside monoids M where the lattice of divisors of the minimal Garside element Δ in M is obtained as a semidirect product of the lattice of divisors of the minimal Garside element δ of a submonoid of M and of a lattice of divisors of Δ prime to the element δ.

The second part of the thesis is sacrificed to the study of the self-distributivity in presence of the idempotency. The self-distributivity is the algebraic identity D: $x(yz) = (xy)(xz)$, and the idempotency is the algebraic identity I: $x = xx$. The LD groupoids, that means the groupoids formed of a set equipped with an operation satisfying the identity LD, have been much studied in the recent years, for instance by Dehornoy [13], Drápal [19], Kepka [34] ou Laver [39]. An *LDI groupoid* is an LD groupoid that satisfies also the identity I. The first problem, when we consider an algebraic identity or a family of algebraic identities, is the *word problem*, defined as the problem to algorithmically decide whether two abstract expressions written with variables and an operation (we say simply two *terms*) are or not equal *modulo* the considered identities. To solve this problem means to describe explicitely the free structures of the equational variety defined by these identities.

In the case of the identity LD, the question has been positively solved in [12], and there exist actually more methods for find out if two terms are or not LD-equivalents. In the case of LDI, that means when we add the idempotency to the self-distributivity, the question is open—and still remains. The results established in this thesis can be seen as partial results in the direction to a solution of the word problem of LDI, which remains to find.

Chapter 4 introduce basic notions needed to study LDI, that means to describe the equivalence relation on terms induced by the identities LD and I. The leading idea is to follow the methods developped in [12] for the case of LD, and to seek to extend them to the case of LDI—which is *not* an immediate task. It seems to be natural to introduce a variation ot the idempotency, namely the identity LI: $(x \cdot x) \cdot y = x \cdot y$, and to make parallelly the study for LDI and for LDLI.

One of the results known for LD is that each class of LD-equivalence has a structure of lattice type (wheter it is a real lattice remains a conjecture in the general case): we introduce an oriented notion of *LD-expansion* raffining the LD-equivalence, and one of the main technical results is the result of *confluence* which affirms that two terms are LD-equivalents if and only if they have a common LD-expansion. This result has been shown also in the case of LDI by Larue in [38]. We remake the proof here and we show a similar result for LDLI:

Proposition 4.12 *Two terms are LDLI-equivalent if and only if they have a common LDLI-expansion.*

The problem for solving the word problem of a family of identities, for instance LDI or LDLI, is to know to show that two terms are *not* equivalent: in fact, if two terms are equivalent, we can always establish it systematically enumerating all the terms equivalent to the first one and watching if the second one appears on the list. There exist two methods for showing that two terms t, t' are not equivalent: the semantic method that consist of constructing an example of a groupoid S satisfying the considered itentities such that t and t' have different evaluations in S, and the syntactic method, consisting to find purely formel criteria showing that t and t' cannot be equivalent. For instance, in the case of LDI, both the identities LD and I conserve the rightmost and the leftmost variables and therefore, if these two variables in t and t' do not coincide, it is impossible for t and t' to be LDI-equivalent. What we would like to, is to define sufficiently fine criteria to separate all the nonequivalent terms, and this would be a solution of the word problem. In the case of LDI (and of LDLI), we do not have a such criterion but we show a partial result enabling to syntactically separate terms we did not know to separate till now. The statement use a certain notion of *cut* of a term.

Proposition 4.29 *If two terms t, t' are LDI-equivalent, each cut of t' belongs to the set* $\mathrm{Cut}(t)$.

We give nontrivial examples of an application of this criterion using a notion of weight on the variables of a term.

In Chapter 5, we study the geometry monoids of the identities LDI and LDLI. For each family of algebraic identities, there exists a monoid that describes the associated equivalence relation on the terms as orbits under the action of a certain monoide of operators [15]. In the case of the associativity, we obtain (essentielly) the Thompson's group F [10], and in the one of the associativity and of the commutativity, we obtain (essentielly) the Thompson's group V [7]. In the case the self-distributivity LD, the geometry monoid is an extension of the Artin's group B_∞, and it is the study of this monoid that enabled to solve the word problem of LD and many connected questions [12]. It is hence natural to study the geometry monoids of the families of identities LDI and LDLI too, particulary hoping to solve the word problems. In the case of LD, the solution comprises four stages, namely the introduction of the syntactic monoid—a monoid satisfying the main relations of the geometry monoid; the proof of the confluence in the syntactic monoid, which means to show that each element of the fraction group of the syntactic monoid can be written in the form uv^{-1} where u and v belong to the monoid; the solution of the word problem of the syntactic monoid; and the construction of an injective mapping from classes of LD-equivalence to the syntactic group. We succeed to complete the first two stages of this approach in the case of LDI and the first three stages in the case of LDLI. The main results of this chaptre are:

Proposition 5.41 *Each element of the fraction group of the syntactic monoid of LDI (respectively of LDLI) can be written as uv^{-1} where u and v belong to the positive syntactic monoid of LDI (respectively of LDLI).*

Proposition 5.46 *The syntactic monoid of LDLI is left cancellative, and each pair of elements admit a right least common multiple and a left greatest common divisor and the word problem of this monoid can be solved by the returning of words.*

Chaptre 6 contains results about the LDLI groupoids, that means about the groupoids satisfying both the identities LD and LI. We describe a decomposition of the LDLI groupoids that enables to reconstruct the LDLI groupoids starting with LDI groupoids and right constant groupoids, that means groupoids satisfying the identity $x \cdot z = y \cdot z$. We use this construction to construct the free LDLI groupoids starting with the free LDI groupoids and we deduce the following result:

Proposition 6.8 *Two terms are LDLI-equivalent if and only if they are LDI-equivalent ant they have the same right height, i.e. the same length of the rightmost branch when the terms are seen as binary trees.*

An easy corollary is that a word problem solution of LDLI would give a word problem solution for LDI, and vice versa. Another application of the proposition is that the equational variety generated by LDLI is the smallest variety that includes both the variety generated by LDI and the one generated by the identity $x \cdot z = y \cdot z$.

I Semidirect products of lattices

1 Semidirect product of lattices

Definition: Let L be a lattice and let θ be a congruence on L. We say that the congruence θ on L is *isoform* if the classes of θ are mutually isomorphic sublattices.

Proposition 1.4: *Let K, H be two lattices and let $\varphi, \psi : K \times K \to H^H$ be two mappings* *satisfying the following conditions:*

$$\varphi_{k,k} = \psi_{k,k} = \mathrm{id}_H, \tag{1.6}$$
$$\varphi_{k,k'\vee k''} = \varphi_{k\vee k',k''} \circ \varphi_{k,k'}, \tag{1.7}$$
$$\psi_{k,k'\wedge k''} = \psi_{k\wedge k',k''} \circ \psi_{k,k'}, \tag{1.8}$$
$$h \leqslant \psi_{k,k\wedge k'} \circ \varphi_{k\wedge k',k}\,(h), \tag{1.9}$$
$$h \geqslant \varphi_{k,k\vee k'} \circ \psi_{k\vee k',k}\,(h), \tag{1.10}$$
$$\varphi_{k,k'}(h \vee h') = \varphi_{k,k'}(h) \vee \varphi_{k,k'}(h'), \tag{1.11}$$
$$\psi_{k,k'}(h \wedge h') = \psi_{k,k'}(h) \wedge \psi_{k,k'}(h'). \tag{1.12}$$

then the set $K \times H$ with the operations $\sqcup$, $\sqcap$, defined by

$$(k_1, h_1) \sqcup (k_2, h_2) = (k_1 \vee k_2,\ \varphi_{k_1,k_2}(h_1) \vee \varphi_{k_2,k_1}(h_2)), \tag{1.17}$$
$$(k_1, h_1) \sqcap (k_2, h_2) = (k_1 \wedge k_2,\ \psi_{k_1,k_2}(h_1) \wedge \psi_{k_2,k_1}(h_2)), \tag{1.18}$$

forms a lattice.

Definition: The lattice constructed in Proposition 1.4 is called a *semidirect product* of lattices K and H, and denoted by $K \ltimes_\psi^\varphi H$.

Proposition 1.8: *Suppose that L is equal to $K \ltimes_\psi^\varphi H$. Then there exists a congruence θ on L such that K is the factor lattice and all equivalence classes of θ are isomorphic to H.*

Proposition 1.9: *Let L be a lattice admitting an isoform congruence θ. Let K be the factor lattice $L/_\theta$ and let H be one of the congruence classes. Suppose that both the following conditions are fulfilled:*

$$\begin{gathered}\text{the set } \{h' \in H;\ \exists\, k, k' \in K:\ (k,h) \leqslant (k',h')\}, \text{ denoted by } E_h,\\ \text{is lower bounded for each } h \text{ in } H,\end{gathered} \tag{1.23}$$
$$\begin{gathered}\text{the set } \{h' \in H;\ \exists\, k, k' \in K:\ (k,h) \geqslant (k',h')\}, \text{ denoted by } E^h,\\ \text{is upper bounded for each } h \text{ in } H.\end{gathered} \tag{1.24}$$

Than there exist mappings φ and ψ satisfying the conditions of Proposition 1.4 such that the lattice L is isomorphic to the lattice $K \ltimes_\psi^\varphi H$.

Proposition 1.12: *Let K, H be two lattices and let φ be a mapping from $K \times K$ to H^H that satisfies $\varphi_{k,k} = \mathrm{id}_H$, for each k in K, and Conditions (1.7) and (1.11), and also, for all k_1, k_2 in K, the condition*

$$\text{the set } \varphi_{k_1,k_2}^{-1}\Big((h]\Big) \text{ owns a smallest element.} \tag{1.30}$$

Then the set $K \times H$ equipped with the order $\leqslant$, defined by

$$(k,h) \leqslant (k',h') \iff (k \leqslant k') \text{ et } (\varphi_{k,k'}(h) \leqslant h'), \tag{1.27}$$

is a lattice.

Proposition 1.13: *(i) Let K, H be two join-semilattices an let φ be a mapping from $K \times K$ to* $\mathrm{End}(H)$ *that satisfies $\varphi_{k,k} = \mathrm{id}_H$, for each k in K, and Condition* (1.11)*. Then the set $K \times H$ equipped with the order $\leqslant$ defined by* (1.27) *is a join-semilattice.*
(ii) Moreover, if K, H are complete and the mapping φ satisfies Condition (1.30) *then $(K \times H, \leqslant)$ is a complete lattice.*

Proposition 1.15: *Let K, H be two join-semilattices and let φ be a mapping from $K \times K$ to* $\mathrm{End}(H)$ *such that $K \ltimes^{\varphi} H$ exists.*
(i) If every interval of K is of finite length then the mapping φ is uniquelly described by the mappings $\varphi_{k,k'}$, with k', an immediate successor of k in K.
(ii) If every interval of H if of finite lenght then we have, for each k' an immediate successor of k in K,

$$\varphi_{k,k'}(h) = \varphi_{k,k'}\big(\min\{h' \in H : \ (h' \geqslant h) \text{ and } ((k', \varphi_{k,k'}(h')) \text{ succeeds immed. } (k, h')))\}\big).$$

Definition: Let $L_1, L_2, \ldots, L_\kappa$ be lattices, for an ordinal κ. We denote by $\sum_{i<\kappa} L_i$ the *ordinal sum* of lattices, defined as the disjoint sum of sets L_i equipped with th ordre $\leqslant$:

$$a \leqslant b \text{ in } L \Longleftrightarrow ((a, b \in L_i) \text{ and } (a \leqslant_{L_i} b)) \text{ or } ((a \in L_i, b \in L_j) \text{ et } (i < j)).$$

Proposition 1.16: *Let L be a lattice and let θ be a nontrivial congruence on L. Then L embeds into a semidirect product of $L/_\theta$ and of an ordinal sum of congruence classes and one element sets. This embedding extends the congruence θ into the canonical congruence of the semidirect product.*

Definition: We denote by $\mathcal{SD}$ the smallest class of lattices that contains the two element lattice and that is closed under sublattices, under semidirect products and under isomorphic images.

Proposition 1.25: *Let L be a finite lattice. Then the following conditions are equivalent:*
(i) the lattice L belongs to $\mathcal{SD}$;
(ii) the lattice L do not contain any sublattice that maps onto a simple lattice;
(iii) the lattice L belongs to the smallest class that contains the two element lattice and that is closed under sublattices, under short exact sequences and under isomorphic images;
(iv) the lattice L belongs to the gratest class of lattices, closed under sublattices and under homomorphic images that does not contain any simple lattice.

Proposition 1.28: *The smallest quasivariety of lattices that contains the two element lattice and that is closed under semidirect product is not a variety.*

2 Construction of the weak order of Coxeter groups

Definition: A *Coxeter graph* $\Gamma = (S, A)$ is a finite nonoriented graph with edges (s,t) in A labelled by a number $m_{s,t}$ of the set $\{3, 4, \ldots, \infty\}$. We say that a group W is a *Coxeter group* associated to the graph of Coxeter $\Gamma = (S, A)$ if it admits the presentation

$$\begin{aligned} \langle\, S;\ s^2 = 1, (st)^2 = 1 &\quad \text{for } (s,t) \notin A, \\ (st)^{m_{s,t}} = 1, &\quad \text{pour } (s,t) \in A \text{ and } m_{s,t} < \infty \,\rangle. \end{aligned} \tag{2.1}$$

We say that the pair (W, S) is a *Coxeter system* if W is a groupe and if there exists a graph of Coxeter $\Gamma = (S, A)$ to which the group W is associated. For each element g of W, we define the *lenght* $\ell(g)$ as the minimal length of a sequence $s_1, s_2, \ldots, s_k$, with s_i in S, satisfying $g = s_1 s_2 \cdots s_k$. The word $s_1 s_2 \cdots s_k$ is then called a *reduced expression.*

Definition: Let W be a Coxeter group. For g, h in W, we write $g \preccurlyeq h$ if and only if we have $\ell(g) + \ell(g^{-1}h) = \ell(h)$. This relation is called the *weak order* of the group W.

Proposition 2.2: *Let W be a Coxeter group. Then the set $(W, \preccurlyeq)$ is a meet-semilattice and the element 1 is its smallest element. If W is finite then $(W, \preccurlyeq)$ is a lattice.*

Definition: Let (W, S) be a Coxeter system and let J a subset of S. We denote by W_J the group generated by J and by W^J the left coset of minimal representatives relative to W_J.

Theorem 2.10: *Let (W, S) be a Coxeter system and let J be a subset of S. Then the meet-semilattice $(W, \preccurlyeq)$ is isomorphic to a semidirect product of the semilattices $(W_J, \preccurlyeq)$ and $(W^J, \preccurlyeq)$.*

Theorem 2.12: *Let (W, S) be a Coxeter system with W finite and let J be a subset of S. Then the lattice $(W, \preccurlyeq)$ is isomorphic to a semi-direct product of the lattices $(W_J, \preccurlyeq)$ and $(W^J, \preccurlyeq)$.*

Proposition 2.19: *Let (W, S) be a Coxeter system and let J be a subset of S. Then the operation of meet in the semilattice $(W, \preccurlyeq)$ is determined using the operations of meet in the semilattices W_J and W^J by the formula:*

$$(k, h) \wedge (k', h') = (k \wedge k', \psi_{k,k'}(h) \wedge \psi_{k',k}(h')). \tag{2.4}$$

where the mapping ψ is expressed by an explicit algorithme.

3 Lattices of divisibility

Definition: Let $\Gamma = (\Sigma, A)$ be a Coxeter graph. The *Artin group* associated to Γ is the group presented by the group presentation

$$\langle\, \Sigma;\ [\sigma,\tau\rangle^{m_{\sigma,\tau}} = [\tau,\sigma\rangle^{m_{\sigma,\tau}} \quad \text{pour } m_{\sigma,\tau} < \infty \,\rangle. \tag{3.1}$$

The *Artin monoid* associated to Γ is the monoid presented by the monoid presentation (3.1). An Artin group or an Artin monoid is said *of spherical type* if the associated Coxeter graph defines afinite Coxter group. We say that an Artin group or an Artin monoid is *irreducible* if its Coxeter graph is connected.

Definition: Let M be a monoid. We say that an element a of M is an *atom* if the relation $a = bc$, for b, c in M, implies $b = 1$ or $c = 1$ We say that the monoid M is *atomic* if the upper bound $\|a\|$ of lengths of decompositions of the element a as a product of atoms is finite for each a in M.

For the order by left divisibility on an Artin monoid of spherical type, there exists a supremum $a \vee b$ and an infimum $a \wedge b$. We define the operation $\backslash$ of the *left residue* by

$$a \vee b = a(a\backslash b) = b(b\backslash a) \tag{3.2}$$

Definition: Let M be an atomic monoid. We say that an element δ of M is *balanced* if the set of left divisors of δ coincide with the set of right divisors of δ. This set is denoted by $M(\delta)$.
An element δ of M is called a *Garside element* if it is balanced and $M(\delta)$ generates M.
A Garside element is called a *minimal Garside element* of M if it is the smallest among all Garside elements of M, with respect to the left divisibility.

Proposition 3.8: *Let M be an irreducible Artin monoid of spherical type that has at least two atoms. We denote by Δ its minimal Garside element and by L the left divisibility lattice of divisors of Δ^k, for $k > 1$. Let θ be a nontrivial congruence on L. If two elements a, b of L are θ-equivalent then the two following conditions are fullfilled:*

$$\{c \in L;\ c \preccurlyeq a \text{ and } \|c\| \leqslant k\} = \{c \in L;\ c \preccurlyeq b \text{ and } \|c\| \leqslant k\}, \tag{3.3}$$
$$\{c \in L;\ a \preccurlyeq c \text{ and } \|c\backslash\Delta^k\| \leqslant k\} = \{c \in L;\ b \preccurlyeq c \text{ and } \|c\backslash\Delta^k\| \leqslant k\}. \tag{3.4}$$

Proposition 3.11: *Let M be an irreducible Artin monoid of spherical type that has at least two atoms. Then the lattice $(M, \preccurlyeq)$ is simple.*

Definition: A monoid M is called a *Garside monoid* if it satisfies the following four conditions:
(i) The monoid M is atomic with a finite number of atoms.

(ii) The monoid M is cancellative.
(iii) Each pair of elements of M admits a right lcm and a left gcd.
(iv) There exists a Garside element in M.

Definition: Let M be a Garside monoid and let Δ be its minimal Garside element. Let δ be a balanced divisor of Δ in M. We denote by M_δ the monoid generated by the set of divisors of δ. If we have $M(\delta) = M(\Delta) \cap M_\delta$ then the monoid M_δ is called a *parabolic subomonoid* of M.

Definition: Let M be a Garside monoid and let Δ be its minimal Garside element. Let M_δ be a parabolic submonoid of M. We say that an element a of $M(\Delta)$ is *δ-reduced* if we have $a \wedge \delta = 1$. The set of all the δ-reduced elements is denoted by M^δ. We denote by d_δ the greatest element of M^δ if such an element exists.

Theorem 3.18: *Let M be a Garside monoid and let Δ be its minimal Garside element. Let M_δ be a parabolic submonoid of M. Suppose that the element d_δ exists and that we have $\Delta = \delta d_\delta$. Then the set M^δ is a lattice and the lattice $M(\Delta)$ is isomorphic to the semidirect product of $M(\delta)$ and M^δ.*

Definition: Suppose that $M_1, \ldots, M_n$ are Garside monoids. Let i be the set of the atoms of M_i for i between 1 and n. A *family of functions verifying the residue identities* is defined as a family $\vec{\Theta}$ of functions $\Theta_{ij} : M_i \times M_j \to M_j$ for $1 \leqslant i \neq j \leqslant n$ such that, for each a in M_i, the restriction $\Theta_{ij}(a, \cdot)$ of Θ_{ij} to $\{a\} \times M_j$ is a bijection of M_j, and verifies

$$\begin{aligned} \Theta_{ij}(ab, c) &= \Theta_{ij}(b, \Theta_{ij}(a, c)), \\ \Theta_{ij}(a, cd) &= \Theta_{ij}(a, c)\Theta_{ij}(\Theta_{ji}(c, a), d), \\ \Theta_{jk}(\Theta_{ij}(a, c), \Theta_{ik}(a, e)) &= \Theta_{ik}(\Theta_{ji}(c, a), \Theta_{jk}(c, e)), \end{aligned}$$

for a, b in M_i, c, d in M_j, e in M_k with $1 \leqslant i \neq j \neq k \neq i \leqslant n$. The *crossed product* $\bowtie_i^{\vec{\Theta}} M_i$ is defined as the factor of the free product of M_i under the congruence generated by all the pairs $(x\Theta_{ij}(x, y), y\Theta_{ji}(y, x))$ with x in A_i, y in A_j and $1 \leqslant i < j \leqslant n$. For $n = 2$, we denote the crossed product by $M_1 \bowtie_{\vec{\Theta}} M_2$.

Proposition 3.22: *Suppose that M_1, M_2 are Garside monoids. Let $\vec{\Theta}$ be a family of functions verifying the residue identies. We denote by Δ_i the minimal Garside element of the monoid M_i, for $i = 1, 2$ and by M the monoid $M_1 \bowtie_{\vec{\Theta}} M_2$. Then the lattice of divisors of the minimal Garside element of M is isomorphic to the semidirect product $M(\Delta_1) \ltimes^\varphi M(\Delta_2)$, where the mapping φ is defined as $\varphi_{a,ax}(b) = \Theta_{12}(x, b)$, for each a in $M(\Delta_1)$, each b in $M(\Delta_2)$ and each x an atom of M_1.*

II Left self-distributive idempotent groupoids

4 Identities LD, I, LI and their expansions

Definition: Let t, t' be two terms.
(*i*) We say that t' is a *basic LD-expansion* of the term t if it is obtained from t by replacing a subterm $t_1 \cdot (t_2 \cdot t_3)$ by the term $(t_1 \cdot t_2) \cdot (t_1 \cdot t_3)$.
(*ii*) We say that t' is a *basic I-expansion* of the term t if it is obtained from t by replacing a subterm $t_1$by the term $t_1 \cdot t_1$.
(*iii*) We say that t' is a *basic LI-expansion* of the term t if it is obtained from t by replacing a subterm $t_1 \cdot t_2$ by the term $(t_1 \cdot t_1) \cdot t_2$.
(*iv*) We say that t' is a *basic LDI-expansion* of the term t if t' is either a basic LD-expansion of t or a basic I-expansion of t.
(*v*) We say that t' is a *basic LDLI-expansion* of the term t if t' is either a basic LD-expansion of t or a LI-expansion of t.

Definition: Let the symbol $\square$ mean one of the families LD, I, LI, LDI or LDLI. We say that a term t' is a *k-$\square$-expansion* of the term t if there exists a sequence $t = t_0, t_1, \ldots, t_k = t'$ such that t_i is a basic $\square$-expansion of t_{i-1}, for all i between 1 and k. We say that t' is a *$\square$-expansion* of t (notation $t \xrightarrow{\square} t'$) if there exists k such that t' is a k-$\square$-expansion of the term t.

Definition: Let t_0, t be two terms. The term $t_0 * t$ is defined inductively:

$$t_0 * t = \begin{cases} t_0 \cdot t & \text{when } t \text{ is a variable,} \\ (t_0 * t_1) \cdot (t_0 * t_2) & \text{for } t = t_1 \cdot t_2. \end{cases} \tag{4.4}$$

We can show by induction that the term $t_0 * t$ is obtained from the term t by applying the substitution $x \mapsto t_0 \cdot x$.

Definition: Suppose that t is a term. We define the terms $\partial_{\mathrm{LD}} t$, $\partial_{\mathrm{LDI}} t$ et $\partial_{\mathrm{LDLI}} t$ inductively:

$$\partial_{\mathrm{LD}} t = \begin{cases} t & \text{when } t \text{ is a variable,} \\ \partial_{\mathrm{LD}} t_1 * \partial_{\mathrm{LD}} t_2 & \text{for } t = t_1 \cdot t_2, \end{cases} \tag{4.5}$$

$$\partial_{\mathrm{LDI}} t = \begin{cases} t \cdot t & \text{when } t \text{ is a variable,} \\ \partial_{\mathrm{LDI}} t_1 * \partial_{\mathrm{LDI}} t_2 & \text{for } t = t_1 \cdot t_2, \end{cases} \tag{4.6}$$

$$\partial_{\mathrm{LDLI}} t = \begin{cases} t & \text{when } t \text{ is a variable,} \\ \partial_{\mathrm{LDI}} t_1 * \partial_{\mathrm{LDLI}} t_2 & \text{pour } t = t_1 \cdot t_2. \end{cases} \tag{4.7}$$

Proposition 4.9: *Let the symbol $\square$ mean one of the families LD, LDI or LDLI and let t, t' be two terms. If t' is a basic $\square$-expansion of t then $\partial_{\square} t$ is a $\square$-expansion of t'.*

Proposition 4.12: *Let the symbol ¤ mean one of the families LD, LDI or LDLI and let t and t' be two terms. Then there exists k such that the term $\partial^k_{¤} t$ is a ¤-expansion of t'.*

Definition: An *addresse* is a finite sequence of 0's and 1's. The empty adresse is denoted by $\varnothing$. The set of all the addresses is denoted by $\mathbf{A}$. An adresse α is said *final* if we have $\alpha = 1^p$ for some $p \geqslant 0$.

Definition: Suppose that t is a term. For α an addresse in $\mathbf{A}$, the *subterm of t at α*, or the α-subterm of t, is the term $\mathrm{sub}(t, \alpha)$, specified possibly as:

$$\mathrm{sub}(t,\alpha) = \begin{cases} t & \text{pour } \alpha = \varnothing, \\ \mathrm{sub}(t_1, \beta) & \text{for } \alpha = 0\beta \text{ and } t = t_1 \cdot t_2, \\ \mathrm{sub}(t_2, \beta) & \text{for } \alpha = 1\beta \text{ and } t = t_1 \cdot t_2. \end{cases} \tag{4.11}$$

Definition: Let t be a term. We say that α is an addresse *in* t if the subterm $\mathrm{sub}(t, \alpha)$ exists. In this cas, we say that α is *external* in t if $\mathrm{sub}(t, \alpha)$ is a variable, and is *internal* otherwise. The *skeleton* of t is defined as the set $\mathrm{Skel}(t)$ of all the addresses in t; the *outline* of t is defined as the set $\mathrm{Out}(t)$ of all the external addresses in t.

Definition: For two terms t_1 and t_2, we write $t_1 \sqsubset t_2$ if we have $t_1 = \mathrm{sub}(t_1, 0^p)$ for some $p > 0$. We denote by $t_1 \sqsubseteq t_2$ the same relation witth $p \geqslant 0$. We write $t_1 \sqsubseteq_{¤} t_2$ if there exist terms t_1 and t_2 with $t'_1 \overset{¤}{=} t_1$, $t'_2 \overset{¤}{=} t_2$ and $t'_1 \sqsubseteq t'_2$.

Definition: For a term t and α in $\mathrm{Skel}(t)$, we define the *cut of t at α* as the term $\mathrm{cut}(t, \alpha)$ recursively:

$$\mathrm{cut}(t,\alpha) = \begin{cases} t & \text{pour } \alpha = \varnothing, \\ \mathrm{cut}(t_1, \beta) & \text{for } \alpha = 0\beta \text{ and } t = t_1 \cdot t_2, \\ t_1 \cdot \mathrm{cut}(t_2, \beta) & \text{for } \alpha = 1\beta \text{ and } t = t_1 \cdot t_2. \end{cases} \tag{4.15}$$

Definition: For a term t, we define the sets $\mathrm{Cut}_{\mathrm{LDI}}(t)$ and $\mathrm{Cut}_{\mathrm{LDLI}}(t)$ as the smallest sets of terms satisfying:
- each cut of the term t belongs to $\mathrm{Cut}_{¤}(t)$;
- each term s' ¤-equivalent to a term s in $\mathrm{Cut}_{¤}(t)$ belongs to $\mathrm{Cut}_{¤}(t)$;
- let s, s' be two terms in $\mathrm{Cut}_{\mathrm{LDI}}(t)$; if there exists a term t' in $\mathrm{Cut}_{\mathrm{LDI}}(t)$ such that s is the cut of t at an addresse α and s' is the cut of t at an addresse α' and if we have $\alpha \geqslant \alpha'$, then the term $s \cdot s'$ belongs to $\mathrm{Cut}_{\mathrm{LDI}}(t)$;
- let s, s' be two terms in $\mathrm{Cut}_{\mathrm{LDLI}}(t)$; if there exists a term t' in $\mathrm{Cut}_{\mathrm{LDLI}}(t)$ such that s is the cut of t at a nonfinal addresse α and s' is the cut of t at an addresse α' and if we have $\alpha \geqslant \alpha'$, then the term $s \cdot s'$ belongs to $\mathrm{Cut}_{\mathrm{LDLI}}(t)$.

From now, the symbol ¤ stands for one of the families LDI or LDLI.

Proposition 4.29: *Let t' be a term ¤-equivalent to a term t. Then each cut of t' belongs to the set* $\mathrm{Cut}_{\text{¤}}(t)$.

Proposition 4.34: *The following conditions are equivalent for two terms s and t:*
(i) we have $s \sqsubseteq_{\text{¤}} t$;
(ii) there exists a term t', with $t \overset{\text{¤}}{=} t'$, and α, an addresse of $\mathrm{Skel}(t')$ *satisfying* $\mathrm{cut}(t', \alpha) = s$;
(iii) we have $s \in \mathrm{Cut}(t)$;
(iv) we have $\mathrm{Cut}(s) \subseteq \mathrm{Cut}(t)$.

Definition: Suppose that α and β are two addresses with $\alpha >_{\mathrm{LR}} \beta$. We say that α *covers* β if there exists an addresse γ and a positif integer q satisfying $\alpha = \gamma 1^q$ and $\gamma 0 \sqsubseteq \beta$. Otherwise we say that α *incovers* β and we write $\alpha \gg \beta$.

Definition: Let t be a term. A *cascade* in t is a finite sequence $\vec{\alpha}$ of pairs $((\alpha_1, a_1), \ldots, (\alpha_p, a_p))$ such that $\alpha_1, \ldots, \alpha_p$ is a strictement decreasing sequence of addresses in $\mathrm{Out}(t)$, and $a_1, \ldots, a_p$ are coefficients 0 or 1, for which $a_i = 0$ implies $i = p$ or $\alpha_i \gg \alpha_{i+1}$. We write $\mathrm{Casc}(t)$ for the set of all the cascades in t.

Proposition 4.38: *(i) For every term t, there exists a bijection π_t between the set* $\mathrm{Casc}(t)$ *and the set* $\mathrm{Out}(\partial_{\mathrm{LDI}} t)$. *Suppose that $\vec{\alpha} = ((\alpha_1, a_1), \ldots, (\alpha_p, a_p))$ is a cascade in t. Then we have:*

$$\mathrm{cut}(\partial_{\mathrm{LDI}} t, \pi_t(\vec{\alpha})) = \partial_{a_1} \mathrm{cut}(t, \alpha_1) \cdot \cdots \cdot \partial_{a_p} \mathrm{cut}(t, \alpha_p), \tag{4.23}$$

where ∂_0 means ∂_{LDI} and ∂_1 means ∂_{LDLI}.

Definition: Let t_0 be a term. For $k \geqslant 0$, we say that a term t is a *t_0-$\partial_{\text{¤}}$-normal of degree k* if t is the cut of $\partial_{\text{¤}}^k t_0$ in an addresse α and no cut of $\partial_{\text{¤}}^k t_0$ at an addresse $\beta <_{\mathrm{LR}} \alpha$ is ¤-equivalent to t. Moreover, for $k > 0$, we require that no cut of $\partial_{\text{¤}}^{k-1} t$ is ¤-equivalent to t neither.

Proposition 4.43: *Let t_0 be a term. Then each term t with $t \sqsubseteq_{\text{¤}} t_0$ is ¤-equivalent to a unique t_0-$\partial_{\text{¤}}$-normal term.*

Conjecture 4.44: *Let t_0 be a term and k an integer. Then each cut t of the term $\partial_{\text{¤}}^k t_0$ is a ¤-expansion of the t_0-$\partial_{\text{¤}}$-normal term that is ¤-equivalent to t.*

Proposition 4.45: *Let t and t_0 be two terms satisfying $t \sqsubseteq_{\text{¤}} t_0$. If Conjecture 4.44 is true, then there exists an algorithme to find the t_0-$\partial_{\text{¤}}$-normal term ¤-equivalent to t.*

5 Geometry monoids

Definition: For each addresse α, we define D_α (respectively I_α) as the partial function from the set of terms to the set of terms that sends each term t on its basic LD-expansion (its basic I-expansion) at the addresse α, if it exists.

Definition: Let $\mathbf{A}_{\mathrm{LD}}$ and $\mathbf{A}_{\mathrm{I}}$ be two disjoint copies of the addresse set $\mathbf{A}$. We denote by $\mathbf{A}_{\mathrm{LDI}}$ the set $\mathbf{A}_{\mathrm{LD}} \cup \mathbf{A}_{\mathrm{I}}$ and, for each α in $\mathbf{A}_{\mathrm{LDI}}$, we define DI_α either as D_α if α belongs to $\mathbf{A}_{\mathrm{LD}}$, or as I_α if α belongs to $\mathbf{A}_{\mathrm{I}}$. We also denote by $\mathbf{A}_{\mathrm{LI}}$ the subset of $\mathbf{A}_{\mathrm{I}}$ defined as $\{\alpha \in \mathbf{A}_{\mathrm{I}};\ \exists\gamma : \alpha = \gamma 0\}$ and by $\mathbf{A}_{\mathrm{LDLI}}$ the set $\mathbf{A}_{\mathrm{LD}} \cup \mathbf{A}_{\mathrm{LI}}$.

Definition: The *geometry monoid* of ⌑ is the monoid $\mathcal{G}_{⌑}$ generated by the operators $\mathrm{DI}_\alpha^{\pm 1}$ with α in $\mathbf{A}_{⌑}$ using the composition. Analogically, the *positive geometry monoid* is the monoid $\mathcal{G}_{⌑}^{+}$ generated by the operateors DI_α with α in $\mathbf{A}_{⌑}$.

Definition: For $w = \alpha_1^{e_1} \cdot \cdots \cdot \alpha_p^{e_p}$ in $(\mathbf{A}_{⌑}^{\pm 1})^*$, the operator DI_w is defined as the product $\mathrm{DI}_{\alpha_1}^{e_1} \bullet \cdots \bullet \mathrm{DI}_{\alpha_p}^{e_p}$, where the symbol $\bullet$ means the composition from the left to the right.

Proposition 5.4: *Let t and t' be two terms.*
(i) The terms t and t' are ⌑-equivalent if and only if an operator in $\mathcal{G}_{⌑}$ sends t onto t', that means if we have $t' = t \bullet \mathrm{DI}_w$ for a word w on $\mathbf{A}_{⌑}^{\pm 1}$.
(ii) The term t is a ⌑-expansion of the term t if and only if an operator in $\mathcal{G}_{⌑}^{+}$ sends t onto t', that means if we have $t' = t \bullet \mathrm{DI}_u$ for a word u on $\mathbf{A}_{⌑}$.

Definition: For t a term, we define the *right height* as:

$$\begin{aligned} \mathrm{rht}(t) &= 0 && \text{when } t \text{ is a variable,} \\ \mathrm{rht}(t) &= \mathrm{rht}(t_2) + 1 && \text{for } t = t_1 \cdot t_2. \end{aligned} \tag{5.6}$$

We see that the right height of a term t is the legth of the rightmost branch of t.

Proposition 5.25: *If Conjecture 4.44 is true then two terms t and t' are LDLI-equivalent if and only if they are LDI-equivalent and they have the same right height.*

Definition: The set $\boldsymbol{\mathcal{A}}_{\mathrm{LDI}}$ is defined as the set of symbols $\mathtt{di}_\alpha$, with α in $\mathbf{A}_{\mathrm{LDI}}$. An *LDI-relation* is a pair of words on $\boldsymbol{\mathcal{A}}_{\text{LDI}}$ among the following relations:

$$
\begin{array}{ccl}
(\mathtt{d}_{\gamma 0\alpha}\cdot \mathtt{d}_{\gamma 1\beta}\ ,\ \mathtt{d}_{\gamma 1\beta}\cdot \mathtt{d}_{\gamma 0\alpha}) & & \text{type } \perp\\
(\mathtt{i}_{\gamma 0\alpha}\cdot \mathtt{d}_{\gamma 1\beta}\ ,\ \mathtt{d}_{\gamma 1\beta}\cdot \mathtt{i}_{\gamma 0\alpha}) & & \text{type } \perp\\
(\mathtt{d}_{\gamma 0\alpha}\cdot \mathtt{i}_{\gamma 1\beta}\ ,\ \mathtt{i}_{\gamma 1\beta}\cdot \mathtt{d}_{\gamma 0\alpha}) & & \text{type } \perp\\
(\mathtt{i}_{\gamma 0\alpha}\cdot \mathtt{i}_{\gamma 1\beta}\ ,\ \mathtt{i}_{\gamma 1\beta}\cdot \mathtt{i}_{\gamma 0\alpha}) & & \text{type } \perp\\
(\mathtt{d}_{\gamma 0\alpha}\cdot \mathtt{d}_{\gamma}\ ,\ \mathtt{d}_{\gamma}\cdot \mathtt{d}_{\gamma 00\alpha}\cdot \mathtt{d}_{\gamma 10\alpha}) & & \text{type D0}\\
(\mathtt{d}_{\gamma 10\alpha}\cdot \mathtt{d}_{\gamma}\ ,\ \mathtt{d}_{\gamma}\cdot \mathtt{d}_{\gamma 01\alpha}) & & \text{type D10}\\
(\mathtt{d}_{\gamma 11\alpha}\cdot \mathtt{d}_{\gamma}\ ,\ \mathtt{d}_{\gamma}\cdot \mathtt{d}_{\gamma 11\alpha}) & & \text{type D11}\\
(\mathtt{d}_{\gamma 1}\cdot \mathtt{d}_{\gamma}\cdot \mathtt{d}_{\gamma 1}\cdot \mathtt{d}_{\gamma 0}\ ,\ \mathtt{d}_{\gamma}\cdot \mathtt{d}_{\gamma 1}\cdot \mathtt{d}_{\gamma}) & & \text{type D1}\\
(\mathtt{i}_{\gamma\alpha}\cdot \mathtt{i}_{\gamma}\ ,\ \mathtt{i}_{\gamma}\cdot \mathtt{i}_{\gamma 0\alpha}\cdot \mathtt{i}_{\gamma 1\alpha}) & & \text{type I}\\
(\mathtt{d}_{\gamma\alpha}\cdot \mathtt{i}_{\gamma}\ ,\ \mathtt{i}_{\gamma}\cdot \mathtt{d}_{\gamma 0\alpha}\cdot \mathtt{d}_{\gamma 1\alpha}) & & \text{type DI}\\
(\mathtt{i}_{\gamma 0\alpha}\cdot \mathtt{d}_{\gamma}\ ,\ \mathtt{d}_{\gamma}\cdot \mathtt{i}_{\gamma 00\alpha}\cdot \mathtt{i}_{\gamma 10\alpha}) & & \text{type ID0}\\
(\mathtt{i}_{\gamma 10\alpha}\cdot \mathtt{d}_{\gamma}\ ,\ \mathtt{d}_{\gamma}\cdot \mathtt{i}_{\gamma 01\alpha}) & & \text{type ID10}\\
(\mathtt{i}_{\gamma 10}\cdot \mathtt{d}_{\gamma}\cdot \mathtt{d}_{\gamma 0}\ ,\ \mathtt{d}_{\gamma}\cdot \mathtt{i}_{\gamma 0}) & & \text{type ID10+}\\
(\mathtt{i}_{\gamma 11\alpha}\cdot \mathtt{d}_{\gamma}\ ,\ \mathtt{d}_{\gamma}\cdot \mathtt{i}_{\gamma 11\alpha}) & & \text{type ID11}\\
(\mathtt{i}_{\gamma 1}\cdot \mathtt{d}_{\gamma}\cdot \mathtt{d}_{\gamma 1}\cdot \mathtt{d}_{\gamma 0}\ ,\ \mathtt{d}_{\gamma}\cdot \mathtt{i}_{\gamma}) & & \text{type ID1}\\
(\mathtt{di}_{\gamma\alpha}\cdot \mathtt{i}_{\gamma 1}\cdot \mathtt{d}_{\gamma}\ ,\ \mathtt{di}_{\gamma\alpha}\cdot \mathtt{i}_{\gamma}) & & \text{type C}\\
(\mathtt{d}_{\gamma}\cdot \mathtt{i}_{\gamma 11}\cdot \mathtt{d}_{\gamma 1}\ ,\ \mathtt{d}_{\gamma}\cdot \mathtt{i}_{\gamma 1}) & & \text{type C}\\
(\mathtt{i}_{\gamma 1}\cdot \mathtt{d}_{\gamma}\cdot \mathtt{di}_{\gamma 1\alpha}\ ,\ \mathtt{i}_{\gamma}\cdot \mathtt{di}_{\gamma 1\alpha}) & \lg(\alpha)\geqslant 1 & \text{type C}\\
(\mathtt{i}_{\gamma 1}\cdot \mathtt{d}_{\gamma}\cdot \mathtt{d}_{\gamma}\ ,\ \mathtt{i}_{\gamma}\cdot \mathtt{d}_{\gamma}) & & \text{type C}\\
(\mathtt{i}_{\gamma 1}\cdot \mathtt{d}_{\gamma}\cdot \mathtt{d}_{\gamma 1}\ ,\ \mathtt{i}_{\gamma}\cdot \mathtt{d}_{\gamma 1}) & & \text{type C}
\end{array}
$$

The set $\boldsymbol{\mathcal{A}}_{\text{LDLI}}$ is defined as the set of symbols $\mathtt{di}_\alpha$, with α in $\mathbf{A}_{\text{LDLI}}$. An *LDLI-relation* is an LDI-relation (u, v) such that u and v belong to $\boldsymbol{\mathcal{A}}_{\text{LDLI}}$. For $\Box$, one of the families LDI, LDLI, the relation $\equiv^+_\Box$ is defined as the congruence of the monoid $\boldsymbol{\mathcal{A}}^*_\Box$ generated by the $\Box$-relations, and the relation $\equiv_\Box$ is defined as the congruence of the monoid $(\boldsymbol{\mathcal{A}}^{\pm 1}_\Box)^*$ generated by the $\Box$-relations and by the relations $(\mathtt{di}_\gamma \cdot \mathtt{di}_\gamma^{-1}, \varepsilon)$ and $(\mathtt{di}_\gamma^{-1} \cdot \mathtt{di}_\gamma, \varepsilon)$.

Proposition 5.26: *For u, v in $\mathbf{A}^*_{\text{LDI}}$, the relation $\mathtt{di}_u \equiv^+_{\text{LDI}} \mathtt{di}_v$ implies $\mathrm{DI}_u = \mathrm{DI}_v$.*

Proposition 5.39: *Let u, v be two words on $\mathbf{A}_\Box$. Then there exist words u', v' on $\mathbf{A}_\Box$ satisfying*

$$\mathtt{di}_u \cdot \mathtt{di}_{v'} \equiv^+_\Box \mathtt{di}_v \cdot \mathtt{di}_{u'}$$

Proposition 5.41: *If w is a word on $\mathbf{A}^{\pm 1}_\Box$ such that the domain of DI_w is not empty then there exist words u, v on $\mathbf{A}_\Box$ satisfying $\mathtt{di}_w \equiv_\Box \mathtt{di}_u \cdot \mathtt{di}_{v^{-1}}$.*

Definition: Let A be an alphabete. We say that f is a *complement* on A if f is a partial mapping from $A \times A$ on A^* satisfying $f(x,x) = \varepsilon$, for each x from A, and that $f(x,y)$ exists if $f(y,x)$ exists. We denote $\equiv_f^+$ the relation generated by the relations $(xf(x,y), yf(y,x))$ with (x,y) in the domain of f. The monoid *associated with f on the right* is the monoid A^* factorised by $\equiv_f^+$. We denote by $\equiv_f$ the relation generated by the relations $(xf(x,y), yf(y,x))$ with (x,y) in the domain of f completed by the relations (xx^{-1}, ε) and $(x^{-1}x, \varepsilon)$. The group *associated with f on the right* is the monoid $(A \cup A^{-1})^*$ factorised by $\equiv_f$.

We define the syntactic monoid of LDLI M_{LDLI} as the monoid $(\mathbf{A}_{\text{LDLI}}^{\pm 1})^*$ factorised by the LDLI-relations. We see that the monoid M_{LDLI} is associated on the right with a complement.

Definition: Let w, w' be two words. We say that w is *returnable (to the right)* into w', denoted by $w \curvearrowright w'$, if there exists a sequence of words $w = w_1, \ldots, w_k = w'$ satisfying, for all $i < k$,

$$w_i = w_i' \cdot x_i^{-1} \cdot y_i \cdot w_i'' \qquad \text{and} \qquad w_{i+1} = w_i' \cdot f(x_i, y_i) \cdot f(y_i, x_i)^{-1} \cdot w_i''$$

where x_i and y_i are letters.

Proposition 5.46: *We have $\mathtt{di}_u \equiv_{\text{LDLI}}^+ \mathtt{di}_v$ if and only if we have $\mathtt{di}_v^{-1} \cdot \mathtt{di}_u \curvearrowright \varepsilon$.*

Proposition 5.47: *The monoid M_{LDLI} is left cancelative and the left divisibility order on M_{LDLI} forms a lattice.*

6 Decomposition of LDLI groupoids

Definition: Let G be a groupoid. We define ip_G as the smallest equivalence relation on G that satisfies $(a, a^2) \in ip_G$.

Proposition 6.4: *For each LDLI groupoid G, the relation ip_G is a congruence and, for all a, b, c in G with $(a, b) \in ip_G$, we have $ac = bc$.*

Definition: A *right constant groupoid* is a groupoid satistying the identity $x \cdot z = y \cdot z$.

Proposition 6.8: (i) *Let H be an LDI groupoid and let A_a, with $a \in H$, be a family of pairwise disjoint sets. Suppose that there exists mappings $f_{a,b}$ from A_b to A_{ab}, for all a, b in H. We define the groupoid $B(H, f)$ as the set $\bigcup_{a \in H} A_a$ with the operation $*$ defined by $x * y = f_{a,b}(y)$, for x in A_a and y in A_b. Then the groupoid $B(H, f)$ is LI. Moreover, if the mappings $f_{a,b}$ satisfy the identity*

$$f_{a,bc} \circ f_{b,c} = f_{ab,ac} \circ f_{a,c} \tag{6.4}$$

for all a, b and c in H, then the groupoid $B(H, f)$ is LD.
(ii) Let G be an LDLI groupoid. Then G is isomorphic to $B(G/ip_G, f)$, where we have $f_{\bar{a},\bar{b}} = ac$ and $\bar{a}$ stands for the class of ip_G that contains a.

Proposition 6.13: *All simple nonidempotent LDLI groupoids are finite of primal cardinality: there exists exactly one such groupoid, up to isomorphism, for an odd prime cardinality, and two such groupoids, up to isomorphism, for the cardinality of two.*

Theorem 6.17: *Two terms t and t' are LDLI-equivalent if and only if they are LDI-equivalent and they have the same right height.*

Corollary 6.18: *The variety of LDLI groupoids is the join of the variety of LDI groupoids and of the one of right constant groupoids.*

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