CENTRAL NILPOTENCY OF SKEW BRACES

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Abstract

Skew braces are algebraic structures related to the solutions of the set-theoretic quantum Yang-Baxter equation. We develop the central nilpotency theory for such algebraic structures in the sense of Freese-McKenzie [12] and we compare the universal algebraic notion of central nilpotency with the notion of right and left nilpotency developed in [7].

Introduction

A discrete version of the braid equation was introduced in [9] as the set-theoretical Yang–Baxter equation (YBE). A pair (X, r) where X is a set and $r : X \times X \to X \times X$ is a bijective map, is a *set-theoretical solution to the Yang–Baxter equation* if

$$(id_X \times r)(r \times id_X)(id_X \times r) = (r \times id_X)(id_X \times r)(r \times id_X) \tag{YBE}$$

holds. A solution (X, r) is *non-degenerate* if the map r is defined as

$$r : X \times X \longrightarrow X \times X, \quad (x, y) \mapsto (\sigma_x(y), \tau_y(x)), \tag{1}$$

where σ_x, τ_x are permutations of X for every $x \in X$. The family of non-degenerate solutions have been studied by several authors [10, 13, 15, 20].

Solutions of the (YBE) can be encoded using certain algebraic structures. Concerning *non-degenerate involutive solutions* (i.e. solutions satisfying $r^2 = id_{X \times X}$), Rump introduced the ring-like binary algebraic structures called *(left) braces* in [17]. Non-involutive solutions are captured by *skew (left) braces* and the latter were defined by Guarnieri and Vendramin in [14].

Finding non-degenerate solutions to (YBE) is a task closely related to the classification problem for skew braces. Indeed, on one hand to any non-degenerate solution is associated a permutation group with a canonical structure of a skew brace. On the other hand all the non-degenerate solutions with a given associated skew brace can be described [3]. Thus, the classification of skew braces is the first step to the study of non-degenerate solutions to (YBE).

Rump [17] introduced two radical series for left braces as a direct analogy of the same notions in rings. These series have been then naturally generalized for skew braces in [7] by using an operation $*$ that plays the role of the multiplication in a ring or of commutator in a group and they led to the definition of *left and right nilpotency* (the same topic has been developed for semi-braces in [6]). Such properties turned out to be practical since they are tied with some algebraic properties of the underlying solution of the Yang-Baxter equation [19].

In universal algebra we have a well-established *commutator theory* started by Jonathan Smith [18] in the 70s and then developed by Ralph Freese and Ralph McKenzie [12] to the version now considered to be canonical. In such a theory, the notion of commutators of congruences, nilpotency and solvability are defined for any algebraic structure, however it may be difficult to translate the formal general definitions to a specific context of interest.

In this paper we decided to focus on the universal algebraic definion of nilpotency applied to skew braces; in order to avoid technical definitions, we use a simplified version of commutator theory, namely the commutator theory for *expanded groups*. And since the word “nilpotency” is already well established in skew braces theory, we decided to call the notion coming from universal algebra as *central nilpotency*. The choice of the name is natural as we use a special ideal of skew braces called the *center*. The notion of center and of central nilpotency translate nicely, in particular the center coincide with the notion of *annihilator* as defined in [11]. On the other hand, the operation $*$ itself is not sufficient to define the commutator in skew braces so we do not give a description of the commutator of two ideals since we still lack enough information.

In a recent paper [4] the notion of *strong left nilpotency* has been developed. This notion is in some sense intermediate as displayed in Figure 1 in which the implications between the different notions of nilpotency are collected. In particular, note that central nilpotency is the strongest property among the ones studied in the literature.

$$\begin{tikzcd}
 & & \text{Central nilpotency} \arrow[dl] \arrow[dr] & & \\
 & \text{Strong left nilpotency} \arrow[dl] \arrow[dr] & & \text{Strong nilpotency} \arrow[dl] \arrow[dr] & \\
\text{Nilpotent type} & & \text{Left nilpotency} & & \text{Right nilpotency}
\end{tikzcd}$$

FIGURE 1. Relation between the various notions of nilpotency.

The article is organized as follows: in Section 1 we define skew braces and we recall some basic properties as well as the notions of nilpotency defined elsewhere. In Section 2 we define the center of a skew brace and the lower and the upper central series tied to the center. We then study properties of these series and of central nilpotency defined through the series. In Section 3 we recall the abstract definitions of the center and of the central nilpotency and we prove that they agree with the notions defined in Section 2.

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1. Basic facts about skew braces

Skew braces. A *skew brace* is a set with two binary operations $(A, +, \circ)$ where both $(A, +)$ and $(A, \circ)$ are groups and

$$x \circ (y + z) = (x \circ y) - x + (x \circ z) \tag{2}$$

holds for every $x, y, z \in A$. The mapping defined as

$$\lambda : (A, \circ) \longrightarrow \mathrm{Aut}(A, +), \quad x \mapsto \lambda_x$$

where $\lambda_x(y) = -x + (x \circ y)$, is a group homomorphism between $(A, \circ)$ and $\mathrm{Aut}(A, +)$. Therefore equation (2) can be written as

$$x \circ (y + z) = (x \circ y) + \lambda_x(z). \tag{3}$$

It can be proved that both groups share the same neutral element, we shall denote it by 0. We shall denote by $-x$ the inverse element of x with respect to $+$ and by $\bar{x}$ the inverse element of x with respect to $\circ$. We say that the brace $(A, +, \circ)$ is of *ϕ-type* if the group $(A, +)$ has the property ϕ (e.g. we say that $(A, +, \circ)$ is of nilpotent-type if $(A, +)$ is nilpotent).

Skew braces were introduced in [14] as a tool for constructing set-theoretic solutions of Yang-Baxter equation (in the new terminology braces, introduced by Rump in [17] are skew braces of Abelian type. We stick to such terminology in the rest of the paper).

A *left ideal* of a skew brace $(A, +, \circ)$ is a subgroup H of $(A, +)$ such that $\lambda_x(H) \leq H$ for every $x \in A$ [14]. Note that every characteristic subgroup of $(A, +)$ is a left ideal and every left ideal is a subgroup of $(A, \circ)$. The subgroup

$$\mathrm{Fix}(A) = \{x \in A \,:\, \lambda_y(x) = x, \text{ for all } y \in A\}$$

is clearly a left ideal.

A quotient of a skew brace $(A, +, \circ)$ is a quotient of both the groups $(A, +)$ and $(A, \circ)$. Therefore, any quotient is given by a normal subgroup I with respect to both group structures and that satisfies $x + I = x \circ I$ for every $x \in A$. This last condition is controlled by the λ mappings. Indeed:

$$x + I = x \circ I \Leftrightarrow -x + (x \circ y) = \lambda_x(y) \in I \text{ for every } y \in I. \tag{4}$$

A normal subgroup of both $(A, +)$ and $(A, \circ)$ satisfying (4) is an *ideal* of A. Hence, an ideal is a normal subgroup I of both $(A, +)$ and $(A, \circ)$ such that $\lambda_x(I) \leq I$ for every $x \in A$ (equivalently I is a left ideal which is also a normal subgroup of $(A, +)$ and $(A, \circ)$). For instance, the socle of A defined as $\mathrm{Soc}(A) = \ker(\lambda) \cap Z(A, +)$ is an ideal.

The $*$ operation. For a skew brace $(A, +, \circ)$, following [7, 17], we define the following binary operation

$$x * y = -x + (x \circ y) - y = \lambda_x(y) - y, \tag{5}$$

for every $x, y \in A$. This operation measures the difference between the two operations $+$ and $\circ$ (through the λ mappings). Indeed, note that $x * y = 0$ if and only if $\lambda_x(y) = y$ if and only if $x + y = x \circ y$. The elements in the subset

$$\mathrm{Fix}(A) \cap \ker(\lambda) = \{x \in A \,:\, x * y = y * x = 0, \text{ for all } y \in A\} \tag{6}$$

are the elements such that $x \circ y = x + y$ and $y \circ x = y + x$ for every $y \in A$. Clearly it is a subgroup of $(A, +)$ and of $(A, \circ)$ and it is trivially invariant under λ_x for every $x \in A$, so it is a left ideal. Moreover,

$$\begin{aligned} (a + x) * (b + y) &= \lambda_{a+x}(b + y) - y - b = \lambda_{a\circ x}(b) + \lambda_{a\circ x}(y) - y - b = \\ &= \lambda_a(b) - b = a * b \end{aligned} \tag{7}$$

for every $x, y \in \mathrm{Fix}(A) \cap \ker(\lambda)$ and $a, b \in A$. Note that $\mathrm{Fix}(A) \cap \ker(\lambda)$ is a left ideal, but it is not an ideal in general (e.g. for `SmallSkewBrace(12,33)` in the database available in GAP [22]).

The $*$ operation satisfies the following identity

$$x * (y + z) = \lambda_x(y + z) - z - y = \lambda_x(y) - y + y + \lambda_x(z) - z - y = x * y + y + x * z - y \tag{8}$$

which is analogous to the equation satisfied by the commutator in groups.

Remark 1.1. Consider a skew-brace $(A, +, +^{op})$. Then $x * y = [x, y^{-1}]_+$. This example shows that the $*$ operation plays a role similar to the element-wise commutator in groups.

Let I, J be subsets of A. We define

$$I * J = \langle i * j,\, i \in I,\, j \in J \rangle_+.$$

If I and J are ideals then, by definition, $I * J \subseteq I \cap J$. It is well known, that a product of two subgroups is a subgroup if and only if the subgroups permute. It is natural to expect a similar behavior for the $*$-product of two ideals.

Lemma 1.2. *Let $(A, +, \circ)$ be a skew brace and I, J be ideals. Then $I * J + J * I$ is a subgroup of $(A, +)$.*

Proof. We need to prove $J*I+I*J\subseteq I*J+J*I$. According to (8), we have

$$\begin{aligned}j_1*i_1+i_2*j_2=-(i_2*(j_1*i_1))+i_2*(j_1*i_1)+j_1*i_1+i_2*j_2-j_1*i_1+j_1*i_1=\\ -(i_2*(j_1*i_1))+i_2*(j_1*i_1+j_2)+j_1*i_1\in I*J+I*J+J*I.\end{aligned}$$

We need also $-(I*J+J*I)\subseteq I*J+J*I$. But clearly

$$-(i_1*j_2+j_2*i_1)=-(j_2*i_1)-(i_1*j_2)\in J*I+I*J\subseteq I*J+J*I.$$

Then easily, by an induction, we get $\langle I*J,J*I\rangle_+=I*J+J*I$. ☐

Proposition 1.3. *Let $(A,+,\circ)$ be a skew brace and I,J be ideals. Then $I*J$, $[I,J]_+$ and $I*J+J*I$ are left ideals of A.*

Proof. Let $i\in I$, $j\in J$, $x\in A$. Then

$$\begin{aligned}\lambda_x(i*j) &= \lambda_x(\lambda_i(j)-j)=\lambda_x\lambda_i(j)-\lambda_x(j)=\\ &= \lambda_{x\circ i\circ\bar{x}}\lambda_x(j)-\lambda_x(j)=\\ &= (x\circ i\circ\bar{x})*\lambda_x(j)\in I*J.\end{aligned}$$

Then $I*J$ is invariant under λ_x for every $x\in A$ and then it is a left ideal. According to Lemma 1.2 $I*J+J*I$ is a subgroup. The sum of left ideals is again a left ideal, then $I*J+J*I$ is a left ideal as well.
Clearly $[I,J]_+$ is a subgroup of $(A,+)$ and $\lambda_x([i,j]_+)=[\lambda_x(i),\lambda_x(j)]_+$ for every $x\in A$. Since I and J are left ideals, then $\lambda_x(i)\in I$ and $\lambda_x(j)\in J$. Therefore $[I,J]_+$ is a left ideal. ☐

The $*$ operation provides a characterization of ideals of a skew braces.

Proposition 1.4. [7, Lemmas 1.8, 1.9] *Let $(A,+,\circ)$ be a skew brace. A normal subgroup I of $(A,+)$ is an ideal if and only if $I*A+A*I\leq I$.*

In [7] two series of left ideals have been defined, for a skew brace A, as

$$A^1 = A,\quad A^{n+1}=A^n*A, \tag{9}$$
$$A^{(1)} = A,\quad A^{(n+1)}=A^{(n)}*A \tag{10}$$

A brace is said to be *right nilpotent* (resp. *left nilpotent*) if $A^{(n)}=0$ (resp. $A^n=0$) for some $n\in\mathbb{N}$. In [19] the following series of left ideals for skew braces was defined:

$$A^{[1]}=A,\quad A^{[n+1]}=\left\langle\bigcup_{i=1}^{n}A^{[i]}*A^{[n+1-i]}\right\rangle_+.$$

It turned out that a skew brace A is left and right nilpotent if and only if $A^{[n]}=0$ for some $n\in\mathbb{N}$. In such case we say that A is *strongly nilpotent* [7, Theorem 2.30].

2. Centrally Nilpotent Skew Braces

We finished the last section with definitions of some nilpotency series. In this section we present yet another definition that mimics the behavior of the nilpotency of groups and we compare this definition with the definitions above.

Let $(A,+,\circ)$ be a skew brace we define the *center* of $(A,+,\circ)$ as $\zeta(A)=\mathrm{Soc}(A)\cap\mathrm{Fix}(A)$. The center of $(A,+,\circ)$ is thus the set

$$\begin{aligned}\zeta(A)&=\{x\in A\,:\,x*y=y*x=[x,y]_+=0,\text{ for every }y\in A\}\\ &=\{x\in A\,:\,x+y=y+x=x\circ y=y\circ x,\text{ for every }y\in A\}.\end{aligned}\tag{11}$$

Note that every subgroup of $(A,+)$ contained in $\zeta(A)$ is an ideal and that $\zeta(A) \leq Z(A,+) \cap Z(A,\circ)$.

This ideal already appeared in the literature: Colazzo, Catino and Stefanelli [11] decided to call this ideal the *annihilator* of a skew brace since it is a counterpart of the annihilator of a ring. On the other hand, Bardakov and Gubarev [4] have found a connection between skew braces and Rota-Baxter groups. Every skew brace can be embedded into a Rota-Baxter group and ideals correspond to normal subgroups. So they name the *center* of a skew brace the ideal corresponding to the center of the Rota-Baxter group. Such an ideal is contained in the ideal defined in (11). This twofold inspiration is not a coincidence as centers of groups and annihilators of rings are two individuals of the same species, species known as the *center* of an expanded group; we shall give the universal definition of the center in Section 3 and we prove in Theorem 3.11 that our definition is a correct specialization of the general definition into the context of skew braces. We stick to the group-like terminology, based on the fact that (11) corresponds to the universal algebraic definition.

Remark 2.1. In a skew brace, the sets $\mathrm{Soc}(A)$ and $\mathrm{Fix}(A)$ can be completely independent: consider, for instance, two abelian groups K and H and let $\alpha : H \to \mathrm{Aut}(K)$. Let us consider two group operations on the set $B = K \times H$: let $(B,+) = K \times H$ and let $(B,\circ) = K \rtimes_\alpha H$; in this way we obtain a brace called a semidirect product of trivial braces. It is easy to compute that $\mathrm{Soc}(B) = K \times \mathrm{Ker}(\alpha)$ and $\mathrm{Fix}(B) = \cap_{h\in H}\mathrm{Fix}(\alpha(h)) \times H$. For instance, if K is an abelian group of an odd order, $H = \mathbb{Z}_2$ and $\alpha(i) = (-1)^i$ then B is a brace with trivial center.

Definition 2.2. A skew brace $(A,+,\circ)$ is said to be *centrally nilpotent of class* n if there exists a chain of ideals

$$0 = I_0 \leq I_1 \leq \ldots \leq I_n = A, \tag{12}$$

such that $I_{j+1}/I_j \leq \zeta(A/I_j)$ for every $0 \leq j \leq n-1$.

There are canonical chains of subgroups that measure the class of central nilpotency:

Definition 2.3. Let A be a skew brace, $n \in \mathbb{N}$ and $I \subseteq A$. We define the *upper central series* of A and the *lower central series* of I as follows:

$$\zeta_0(A) = 0, \qquad \zeta_n(A) = \{x \in A \,:\, x * y,\, y * x\,, [x,y] \in \zeta_{n-1}(A) \text{ for every } y \in A\} \tag{13}$$

$$\Gamma_0(I) = I, \qquad \Gamma_n(I) = \langle \Gamma_{n-1}(I) * A,\, A * \Gamma_{n-1}(I),\, [\Gamma_{n-1}(I), A]_+\rangle_+. \tag{14}$$

In particular $\zeta_1(A) = \zeta(A)$ as defined in (11).

Example 2.4. Let $(A,+,+^{op})$ be a skew brace. Then the upper and lower central series of the skew brace are the same as the upper and lower central series of the group $(A,+)$ since $*$ is the ordinary group commutator here, according to Remark 1.1.

Lemma 2.5. *Let $(A,+,\circ)$ be a skew brace. Then $\zeta_n(A)$ is an ideal and $\zeta_{n+1}(A)/\zeta_n(A) = \zeta(A/\zeta_n(A))$ for every $n \in \mathbb{N}$.*

Proof. If $n = 1$ then $A*\zeta_1(A)+\zeta_1(A)*A = 0$ and then $\zeta_1(A)$ is an ideal, according to Proposition 1.4. Assume by induction that $\zeta_n(A)$ is an ideal. The set $\zeta_{n+1}(A)$ is the preimage of $\zeta(A/\zeta_n(A))$ under the canonical projection onto $A/\zeta_n(A)$ then it is an ideal. □

Lemma 2.6. *Let $(A,+,\circ)$ be a skew brace and I be an ideal of A. Then $\Gamma_n(I)$ is an ideal of A for every $n \in \mathbb{N}$.*

Proof. The statement is true for $n = 0$. Assume by induction that $\Gamma_{n-1}(I)$ is an ideal. Therefore we have that $\Gamma_n(I) \subseteq \Gamma_{n-1}(I)$. By definition of $\Gamma_n(I)$ it follows that $\Gamma_n(I) * A \subseteq \Gamma_{n-1}(I) * A \subseteq \Gamma_n(I)$, $A * \Gamma_n(I) \subseteq A * \Gamma_{n-1}(I) \subseteq \Gamma_n(I)$ and $a + \Gamma_n - a \subseteq a + \Gamma_{n-1} - a \subseteq \Gamma_n(I)$. So $\Gamma_n(I)$ is a normal subgroup of $(A,+)$ and $A * \Gamma_n(I) + \Gamma_n(I) * A \subseteq \Gamma_n(I)$. According to Lemma 1.4 $\Gamma_n(I)$ is an ideal. □

Lemma 2.7. *Let A be a skew brace, $I \subseteq A$ and $n, k \in \mathbb{N}$. Then $\Gamma_n(I) \leq \zeta_k(A)$ if and only if $I \leq \zeta_{n+k}(A)$.*

Proof. Let us proceed by induction on n. If $n = 0$ the statement is trivial. Now we want to prove, for any $n \geq 0$, that $\Gamma_n(I) \leq \zeta_{k+1}(A)$ is equivalent to $\Gamma_{n+1}(I) \leq \zeta_k(A)$. But this is clear from definitions (13) and (14). □

Theorem 2.8. *Let $(A, +, \circ)$ be a skew brace. The following are equivalent:*

(i) *$(A, +, \circ)$ is centrally nilpotent of class n.*
(ii) *$\zeta_n(A) = A$.*
(iii) *$\Gamma_n(A) = 0$.*

Proof. (i) $\Leftrightarrow$ (ii) is due to Lemma 2.5.
(ii) $\Leftrightarrow$ (iii) According to Lemma 2.7, $\Gamma_n(A) = 0 = \zeta_0(A)$ if and only if $A \leq \zeta_n(A)$ □

Corollary 2.9. *Let $(A, +, \circ)$ be a centrally nilpotent skew brace. Then $(A, +, \circ)$ is a strongly nilpotent brace of nilpotent type and $(A, \circ)$ is nilpotent.*

Proof. It follows since $A^n, A^{(n)} \leq \Gamma_n$ for every $n \in \mathbb{N}$ and the ideal $\zeta_1(A)$ is contained in $Z(A, +) \cap Z(A, \circ)$. □

Remark 2.10. Let $(A, +)$ be a perfect group. The skew brace $(A, +, +)$ is strongly nilpotent (indeed $x * y = 0$ for every $x, y \in A$), but it is not centrally nilpotent since $A = [A, A]_+$.

Proposition 2.11. *Let $(A, +, \circ)$ be a strongly nilpotent skew brace of nilpotent type and I be a non trivial ideal of A. Then $\zeta(A) \cap I \neq 0$. In particular, $\zeta(A) \neq 0$.*

Proof. Assume that $(A, +, \circ)$ is both left and right nilpotent. Then $\mathrm{Soc}(A) \cap I \neq 0$ since it is right nilpotent of nilpotent type (see [7, Theorem 2.8]) and $I \cap \zeta(A) = I \cap \mathrm{Soc}(A) \cap \mathrm{Fix}(A) \neq 0$ since it is left nilpotent ([7, Proposition 2.26]). If $I = A$ we have that $\zeta(A) \neq 0$. □

This gives us that finite skew braces are centrally nilpotent if and only if they are left and right nilpotent skew braces of nilpotent type.

Corollary 2.12. *Let $(A, +, \circ)$ be a finite skew brace. The following are equivalent*

(i) *$(A, +, \circ)$ is a strongly nilpotent skew brace of nilpotent type.*
(ii) *$(A, +, \circ)$ is right nilpotent skew brace of nilpotent type and $(A, \circ)$ is nilpotent.*
(iii) *$(A, +, \circ)$ is a centrally nilpotent skew brace.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows by [7, Theorem 4.8] and [7, Theorem 2.20] and the implication (iii) $\Rightarrow$ (i) by Corollary 2.9.
(i) $\Rightarrow$ (iii) According to Proposition 2.11 the series of the ideals in (13) is ascending. Since A is finite, then $A = \zeta_n(A)$ for some $n \in \mathbb{N}$. □

For skew braces of abelian type we can extend Corollary 2.12 to the infinite case. Notice that if A is of abelian type then $\zeta(A) = \ker(\lambda) \cap Fix(A)$.

Proposition 2.13. *Let $(A, +, \circ)$ be a skew brace of abelian type. The following are equivalent:*

(i) *$(A, +, \circ)$ is strongly nilpotent.*
(ii) *$(A, +, \circ)$ is right nilpotent and $(A, \circ)$ is nilpotent.*
(iii) *$(A, +, \circ)$ is centrally nilpotent.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is [19, Theorem 3] and (iii) $\Rightarrow$ (i) is true in general (see Corollary 2.9).
(i) $\Rightarrow$ (iii) Assume that $(A, +, \circ)$ is both right and left nilpotent. Then there exists $n \in \mathbb{N}$ such that $A^{[n]} = 0$ according to [19, Theorem 3]. Let proceed by induction on n.

If $A^{[1]} = 0$ then A is the trivial group and in particular it is a nilpotent brace. Assume that $A^{[n]} = 0$, then $A^{[n-1]} * A = A * A^{[n-1]} = 0$, therefore $A^{[n-1]} \leq \operatorname{Fix}(A) \cap \ker(\lambda) = \zeta(A)$. Hence $(A/\zeta(A))^{[n-1]} = 0$. By induction, $A/\zeta(A)$ is centrally nilpotent and A is centrally nilpotent as well. $\square$

3. A universal algebraic motivation

The motivation why we came up with the definition of the central nilpotency is the notion of nilpotency coming from the universal algebraic commutator theory developed in [12]. The general theory is very technical and therefore we present the notions in the narrower class of expanded groups, a class that contains groups, rings, vector spaces, skew braces etc.

Let us have a set A with several operations $o_1, \dots, o_k$ of arities $\varepsilon_1, \dots, \varepsilon_k$. Let us recall that a *term* on A is either a variable or an expression $o_i(t_1, \dots, t_{\varepsilon_i})$, where $t_1, \dots, t_{\varepsilon_i}$ are terms on A. An *endomorphism* of such a structure is a mapping $\varphi : A \to A$ such that $\varphi(o_i(a_1, \dots, a_{\varepsilon_i})) = o_i(\varphi(a_1), \dots, \varphi(a_{\varepsilon_i}))$, for any i and all $a_1, \dots, a_{\varepsilon_i} \in A$.

Definition 3.1. An *expanded group* is a set A with several operations of several arities, among them a binary operation $+$, a unary operation $-$ and a constant 0 such that $(A, +, -, 0)$ is a group. A n-ary *polynomial* of A is a function $A^n \to A$ such that there exist constants $c_1, \dots, c_k$ and a $n+k$-ary term t such that $f(x_1, \dots, x_n) = t(c_1, \dots, c_k, x_1, \dots, x_n)$. We denote by $\operatorname{Pol}_n(A)$ the set of all n-ary polynomials of A.

Examples of expanded groups are, for instance, rings or vector spaces. In the class of rings, polynomials are the classical polynomial functions. In the class of vector spaces, polynomials are affine functions. Polynomials are used to define commutators.

Definition 3.2. Let A be an expanded group. Let $f(x_1, \dots, x_n) \in \operatorname{Pol}_n(A)$. We say that f is *absorbing* if, for all $1 \leq i \leq n$ and all $a_j \in A$, with $1 \leq j \leq n$, $f(a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n) = 0$. A subset I of an expanded group A is called an *ideal* if it there exists an endomorphism φ of A such that $\varphi(i) = 0$ if and only if $i \in I$. Let I, J be two ideals of A. A *commutator* $[\![I, J]\!]$ of two ideals is defined as the ideal generated by the set

$$\{f(i, j) \ : \ i \in I, j \in J, f \in \operatorname{Pol}_2(A), \ f \text{ is absorbing}\}.$$

In the class of groups, the only binary polynomials with $f(x, 0) = f(0, x) = 0$ are some combinations of (element-wise) commutators and therefore $[\![I, J]\!] = [I, J]$. In the class of rings, the commutator is the smallest ideal containing $I \cdot J + J \cdot I$, since $f = x \cdot y$ is a binary absorbing polynomial. In the class of vector spaces, there are no non-constant absorbing polynomials and therefore the commutator of two subspaces is the trivial subspace.

Proposition 3.3. *Let $(A, +, \circ)$ be a skew brace and I, J be ideals. Then $I*J+J*I+[I, J]_+ \leq [\![I, J]\!]$.*

Proof. Consider the absorbing polynomials $x * y$, $y * x$ and $x + y - x - y$. $\square$

Problem 3.4. Let $(A, +, \circ)$ be a skew brace and I, J be ideals. Is it true that $[\![I, J]\!]$ is the smallest ideal containing $I * J + J * I + [I, J]_+$? Does the equality $[\![I, J]\!] = I * J + J * I + [I, J]_+$ hold?

Let us stop by this problem for a little while. The examples above, namely the commutator of normal groups or the commutator of ring ideals are somewhat misleading since the only proper absorbing polynomials are binary. We mean that there exist absorbing polynomials of higher arities like $[x, [y, z]]$ for groups or xyz for rings but these polynomials are actually nested binary absorbing polynomials. Therefore, when constructing the commutator of two normal subgroups I and J or two ring ideals I and J, we can do it *locally*, that means, we can consider only elements within I or J and the commutator does not depend on the whole expanded group A. This is not a general situation as a class of expanded groups may possess proper ternary absorbing polynomials.

A good example for illustration are loops; they are something like "groups without associativity". Actually they are not expanded groups but they are not far away. Loops have a nontrivial ternary absorbing polynomial, namely $(x + (y + z)) - ((x + y) + z)$ and therefore, when constructing the commutator of two normal subloops I and J, we have to consider, among others, all the elements of form

$$(a + (b + c)) - ((a + b) + c), \qquad a \in I, b \in J, c \in A$$

coming from the binary absorbing polynomials $(x+(y+c))-((x+y)+c)$ where $c \in A$. This means that the commutator of two normal subloops cannot be computed locally and it always depends on the whole loop A in which we compute. See [21] for more information about the commutator theory for loops.

In the case of skew braces we have found no evident proper ternary absorbing polynomial but it does not mean that there is none. Hence a problem which is crucial for solving Problem 3.4 is the following.

Problem 3.5. Are all absorbing polynomials in a skew brace nested binary absorbing polynomials?

For short, we still do not know how to compute the commutator of two ideals in skew braces. An exception is when one of the ideals is the entire skew brace.

Definition 3.6. Let A be an expanded group. We say, that an ideal I is *central* if $[\![A, I]\!] = 0$. We say that A is *abelian* if A is central in A. The *center* of A is the largest central ideal of A.

Example 3.7. We give a few examples of centers of expanded groups:

(i) For a ring R, a two-sided ideal is central if and only if it lies in $\mathrm{Ann}_R(R)$. Therefore, the center of a ring is its annihilator. And a ring is abelian if and only if $R \cdot R = 0$.
(ii) Every vector space is abelian since the commutator of two subspaces is always trivial.
(iii) A prototypical binary absorbing polynomial for Lie algebras is the Lie bracket. Therefore the center of a Lie algebra is the radical of the Lie bracket.

Using Proposition 3.3 we have immediately the following Corollary.

Corollary 3.8. *Let $(A, +, \circ)$ be a skew brace. Then every central ideal of A is contained in $\zeta(A)$.*

In the theory of left braces the word *abelian* is already used for left braces with both the groups abelian, not necessarily isomorphic. Which is actually a much wider class than abelian skew braces in the sense of the commutator theory.

Corollary 3.9. *Let $(A, +, \circ)$ be a skew brace. The following are equivalent:*

(i) *$(A, +, \circ)$ is abelian (in the sense of Definition 3.6),*
(ii) *the operations $+$ and $\circ$ are commutative and they coincide,*
(iii) *$A * A = [A, A]_+ = 0$.*

We have already defined the notion of a *center* in the beginning of our Section 2. Now we shall prove that this definition is conforming with the commutator theory, that means, that $\zeta(A)$ is the center of A in the sense of Definition 3.6. For our proofs concerning the centralizing relation for ideals, we work with terms of skew braces as the terms in the language $\{+, -, *, \overline{()}, 0\}$: it is not hard to see that this language is equivalent since, having a term in the language $\{+, -, \circ, \overline{()}, 0\}$, we can replace every occurrence of $\circ$ using

$$x \circ y = x + (x * y) + y.$$

Actually, this is the path along which years ago two-sided braces evolved from radical rings.

Lemma 3.10. *Let $(A, \cdot, \circ)$ be a skew brace and let $t(x_1, \ldots, x_n)$ a term in which the variable x_1 is not involved in any $*$ operation. Then there exists $k \in \mathbb{Z}$ such that $t(x_1 + z, \ldots, x_n) = t(x_1, \ldots, x_n) + kz$ for every $z \in \zeta(A)$ and every $x_1, \ldots, x_n \in A$.*

Proof. We may assume that the variable x_1 is actually not present in t. In this case, the statement is vacuously true.

Suppose that x_1 is present in t and we prove the lemma by induction on the height of t. Recall that if $z \in \zeta(A)$ then $z \in Z(A,+) \cap Z(A,\circ)$ and $-z = \overline{z}$. If the height is zero, the statement is true. Suppose hence that the height is positive. Hence there are three possibilities:
1) $t = -s$; then

$$t(x_1 + z, \dots, x_n) = -s(x_1 + z, \dots, x_n) = -(s(x_1, \dots, x_n) + kz) = t(x_1, \dots, x_n) - kz.$$

2) $t = \bar{s}$; then

$$\begin{aligned} t(x_1 + k, \dots, x_n) = \overline{s(x_1 + z, \dots, x_n)} = \overline{(s(x_1, \dots, x_n) + kz)} = \overline{(s(x_1, \dots, x_n) \circ kz)} \\ = \overline{kz} \circ t(x_1, \dots, x_n) = t(x_1, \dots, x_n) - kz. \end{aligned}$$

3) $t(x_1, \dots, x_n) = s(x_1, \dots, x_n) + r(x_1, \dots, x_n)$; then

$$\begin{aligned} t(x_1 + z, \dots, x_n) = s(x_1 + z, \dots, x_n) + r(x_1 + z, \dots, x_n) = \\ s(x_1, \dots, x_n) + nz + r(x_1, \dots, x_n) + mz. = t(x_1, \dots, x_n) + (n+m)x. \end{aligned}$$

□

Theorem 3.11. *Let $(A,+,\circ)$ be a skew brace. Then $\zeta(A)$ is the biggest central ideal of A.*

Proof. According to Corollary 3.8, we only need to prove $[\![A, \zeta(A)]\!] = 0$, that means, $f(x,y) = 0$, for any binary absorbing polynomial f. Assume that f is obtained as $f(x,y) = t(x,y,a_1,\dots,a_n)$ for some $n+2$-ary term $t(x,y,z_1,\dots,z_n)$ and constants $a_1,\dots,a_n$.

Consider one of the lowest node in which the operation $*$ appears and involves the variable y. Then you have a subterm which look like this $s(x,y,x_1,\dots,x_n) * r(x,y,x_1,\dots,x_n)$, where s and r are subterms. The subterms $r(x,y,x_1,\dots,x_n)$ and $s(x,y,x_1,\dots,x_n)$ contains just group operations $+$ and inversions involving x, then using Lemma 3.10 there exists $k,m \in \mathbb{Z}$ such that $s(x,y,x_1,\dots,x_n) = s(x,0,x_1,\dots,x_n) + my$ and $r(x,y,x_1,\dots,x_n) = r(x,0,x_1,\dots,x_n) + ky$ for every $y \in \zeta(A)$. Therefore, assuming that $y \in \zeta(A)$ we have

$$\begin{aligned} s(x,y,x_1,\dots,x_n) * r(x,y,x_1,\dots,x_n) &= (s(x,0,x_1,\dots,x_n) + my) * (r(x,0,x_1,\dots,x_n) + ky) \\ &\overset{(7)}{=} s(x,0,x_1,\dots,x_n) * r(x,0,x_1,\dots,x_n). \end{aligned}$$

Hence, if $y \in \zeta(A)$, we can remove y from such a node and we can apply the same argument to the next node $*$ involving the variable y. So we can run through every branch of the term removing y from every $*$ node.

Thus, we can assume that y is not involved in any $*$ operation. Then, by Lemma 3.10, we have

$$f(x,y) = t(x,y,a_1,\dots,a_n) = t(x,0,a_1,\dots,a_n) + ny = f(x,0) + ny$$

for some $n \in \mathbb{Z}$, whenever $y \in \zeta(A)$. Thus, since f is absorbing we have

$$f(x,y) = f(x,0) + ny = 0 + ny = f(0,0) + ny = f(0,y) = 0,$$

for every $y \in \zeta(A)$ and $x \in A$. □

According to Theorem 3.11 and Theorem 2.8 the series defined in (13) and (14) coincide with the upper and lower central series in the sense of commutator theory.

For extended groups there exists a standard construction named *the central extension* given in Proposition 7.1 of [12] and such a construction correspond to central congruences in the sense of [12] in a wider class of algebraic structures, including extended groups. In the context of braces, such a construction has been called *Hochschild product of skew braces*, [11]. The main result of such a paper is actually just a particular case of [12, Proposition 7.1], since the annihilator of a skew brace coincides with the center in the sense of commutator theory.

Theorem 3.12. [11, Theorem 24] *Let B be a skew brace. Then $B \cong (B/\zeta(B) \times \zeta(B), +, \circ)$, where*

$$\begin{aligned}
(x,a)+(y,b) &= (x+y, a+b+\tau(x,y)) && (15)\\
(x,a)\circ(y,b) &= (x\circ y, a+b+\theta(x,y)) && (16)\\
\theta(x,y+z)+\tau(y,z) &= \theta(x,y)+\theta(x,z)-\tau(x,-x+x\circ z)+\tau(x\circ y,-x+x\circ z), && (17)
\end{aligned}$$

for every $x,y,z \in B/\zeta(B)$, $a,b,c \in \zeta(B)$, where $\theta : (B/\zeta(B),+)^2 \to \zeta(B)$ is an abelian group cocycle and $\phi : (B/\zeta(B),\circ)^2 \to \zeta(B)$ is a group cocycle.

We are going to show a construction of braces, similar to the construction given in [8, Section 9] using bilinear mappings. Actually, this construction is a special case of the central extension given in Theorem 3.12. Recall that bilinear mappings are special cases of group cocycles.

Corollary 3.13. *Let $(H,+,\circ)$ be a centrally nilpotent skew brace of length n, K be an abelian group and $\theta : (H,+)\times(H,+) \longrightarrow (K,+)$ be a bilinear map. Then $(K,\times K,+,\circ)$ where*

$$\begin{aligned}
(x,y)+(z,t) &= (x+y, z+t)\\
(x,y)\circ(z,t) &= (x\circ y, z+t+\theta(x,y))
\end{aligned}$$

for every $x,z \in H$ and $y,t \in K$ is a centrally nilpotent brace of length less or equal to $n+1$.

Proof. The construction of A is the Hochschild product of $(H,+,\circ)$ and $(K,+,+)$ for $\tau = 0$ and so A is a skew brace. Since $\{1\}\times K \leq \zeta(A)$ and $A/(\{1\}\times K) = (H,+,\circ)$ then A is centrally nilpotent of length at most $n+1$. □

Example 3.14. Let E,F,A be abelian groups and $\omega : E\times F \longrightarrow A$ be a bilinear mapping. Let $\mathbb{H}(\omega) = (E\times F\times A,\circ)$ is the generalized Heisenberg group defined as

$$(e_1,f_1,a_1)\circ(e_2,f_2,a_2) = (e_1+e_2, f_1+f_2, a_1+a_2+\omega(e_1,f_2)),$$

for every $e_1,e_2 \in E$, $f_1,f_2 \in F$ and $a_1,a_2 \in A$. Actually, this operation is usually displayed in a matrix-like style, namely

$$\begin{bmatrix}1 & e_1 & a_1\\ 0 & 1 & f_1\\ 0 & 0 & 1\end{bmatrix} \circ \begin{bmatrix}1 & e_2 & a_2\\ 0 & 1 & f_2\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & e_1+e_2 & a_1+a_2+\omega(e_1,f_2)\\ 0 & 1 & f_1+f_2\\ 0 & 0 & 1\end{bmatrix}.$$

Then $(E\times F\times E,+,\circ)$ where $(E\times F\times A,+)$ is the direct product of the three groups and $(E\times F\times A,\circ) = \mathbb{H}(\omega)$ is a nilpotent brace of length 2. See [5] for further details about Heisenberg groups.

At the end we present one more definition of nilpotency. We have seen that there exist several definitions that generalize the notion of nilpotent groups or nilpotent rings. Each of them has its advantages and each of them is of different strength. This is not unusual; even on the level of universal algebra the standard notion of nilpotency is not the unique universal generalization of nilpotency – there exists a stronger generalization called the supernilpotency.

Definition 3.15. Let A be an expanded group. We define *higher commutators* as follows: let $I_1,\dots,I_k$ be ideals of A then $[\![I_1,\dots,I_k]\!]$ is the ideal generated by

$$\{f(x_1,\dots,x_k) \ : \ x_i \in I_i, f \in \mathrm{Pol}_k(A),\ f \text{ absorbing}\}.$$

We say that A is *supernilpotent of class k* if $[\![\underbrace{A,A,\dots,A}_{k+1 \text{ times}}]\!] = 0$.

Another reformulation of the definition is that A is k-supernilpotent if and only if every $k+1$-ary absorbing polynomial is constant. As we have already mentioned, in the case of group or rings, essentially the only absorbing polynomials are nested binary absorbing polynomials and therefore the notions of nilpotency and of supernilpotency coincide for groups [1] or for rings [16]. In general, however, the supernilpotency is a different property and there are examples of structures that are nilpotent but not supernilpotent [21]. The supernilpotency is generally a stronger property but in the case of skew braces it may be of equal strength.

Theorem 3.16. [2, Corollary 6.15] *Let A be a skew brace which is supernilpotent of class k. Then A is centrally nilpotent of class at most k.*

A converse implication does not hold for expanded groups in general. The class of skew braces is closely related to rings or groups and therefore it is well possible that supernilpotency and central nilpotency are the same properties for them. But again, this question is related to Problem 3.5.

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