# SUBDIRECTLY IRREDUCIBLE MEDIAL QUANDLES 

PŘEMYSL JEDLIČKA, AGATA PILITOWSKA, AND ANNA ZAMOJSKA-DZIENIO


#### Abstract

We classify subdirectly irreducible medial quandles. We show that in the finite case they are either connected (and therefore affine) or reductive. Moreover, we give an explicit description of all subdirectly irreducible reductive medial quandles.


## 1. Introduction

A binary algebra $(Q, \cdot)$ is called a quandle if the following conditions hold, for every $x, y, z \in Q$ :

- $x x=x$ (we say $Q$ is idempotent),
- $x(y z)=(x y)(x z)$ (we say $Q$ is left distributive),
- the equation $x u=y$ has a unique solution $u \in Q$ (we say $Q$ is a left quasigroup).

A quandle $Q$ is called medial if, for every $x, y, u, v \in Q$,

$$
(x y)(u v)=(x u)(y v) .
$$

A prototypic example is the class of affine quandles: given an abelian group $(A,+)$ with an automorphism $f$, let $\operatorname{Aff}(A, f)$ denote the quandle over the set $A$ with the operation $x * y=(1-f)(x)+f(y)$. Alternatively, affine quandles can be regarded as reducts of $\mathbb{Z}\left[x, x^{-1}\right]$-modules.

This paper continues the research on medial quandles we started in [7], and we refer to its introduction for motivating remarks. Our main result in [7] states that all medial quandles are built from affine pieces using a heterogeneous affine structure, called affine mesh, where the affine pieces correspond to the orbits of the multiplication group.

The aim of this paper is to further develop the structure theory of medial quandles (mainly in the reductive case) and to apply the theory to classify (finite) subdirectly irreducible medial quandles.

An algebra is called simple if it has no non-trivial homomorphic images, or, equivalently, no nontrivial congruence relations (i.e., equivalence relations invariant with respect to the operations). Finite simple quandles were classified independently in $[1,9]$. Since the orbit decomposition provides a congruence, simple quandles with more than two elements must be connected, hence, in the medial case, affine. As a special case of the classification, we obtain that a finite medial quandle $Q$ is simple if and only if $Q \simeq \operatorname{Aff}\left(\mathbb{Z}_{p}^{k}, M\right)$ where $p$ is a prime and $M$ is the companion matrix of an irreducible monic polynomial in $\mathbb{F}_{p}[x]$.

Classification of simple objects in a class $\mathcal{C}$ allows to investigate the properties of algebras in $\mathcal{C}$. However, this requires a good understanding of extensions, such as in groups, and this is not always the case, for instance in quandles. For that reason, universal algebra developed a stronger type of representation.

An algebra $A$ is called a subdirect product of algebras $S_{i}, i \in I$, if it embeds into the direct product $\prod_{i \in I} S_{i}$ in a way that every projection $A \rightarrow S_{i}$ is onto. An algebra $S$ is called subdirectly irreducible (SI) if it admits no non-trivial subdirect representation. Birkhoff's theorem says that every algebra in a variety $\mathcal{V}$ embeds in a subdirect product of SI algebras from $\mathcal{V}$. Therefore, knowledge of SI

[^0]algebras in a given variety provides a powerful tool, avoiding the need to understand congruences and extensions. Another Birkhoff's theorem says that an algebra $S$ is subdirectly irreducible if and only if the intersection of all non-trivial congruences, called the monolith congruence, is nontrivial, thus providing an easy-to-use criterion of subdirect irreducibility (in particular, every simple algebra is SI). See [2, Section 3.3] for details.

Here we summarize previous work on subdirectly irreducible medial quandles. On one hand, there are two papers which deal with special cases: Roszkowska [13] gave an explicit construction of all SI medial quandles that are involutory (2-symmetric), and Romanowska and Roszkowska [11] did the same for the 2 -reductive ones. In a broader perspective, Kearnes [6] studies SI algebras in a larger class of modes [12], idempotent algebras with a commutative clone of term operations. Kearnes classifies SI modes according to the algebraic properties of blocks of their monolith. An SI mode $S$ has precisely one of the following three types:

- the set type: all monolith blocks are trivial algebras (i.e. all operations are projections);
- the quasi-affine type: all monolith blocks are non-trivial algebras and embed into an affine algebra;
- the semilattice type: the mode $S$ (and thus each monolith block) has a semilattice term.

The semilattice type is well understood [5] but cannot appear in quandles. The properties of quasiaffine algebras often reduce to module-theoretical questions, see e.g. Section 4. But very little is known about the set type SI modes. A notable exception is [14], a classification of 2-reductive SI modes (they are all of set type). Our paper fills partially the gap in another particular class of modes, namely medial quandles, focusing on reductive medial quandles that are all of set type. Our Theorem 6.12 constructs all reductive SI modes and gives actually a complete classification of all finite SI medial quandles of set type. It is also interesting to note that classification of non-idempotent SI racks uses substantially different techniques, see $[8,15]$.

The paper is organized as follows. In Section 2 notions and results from [7] on representing medial quandles as sums of affine meshes are recalled. Each medial quandle can be constructed from abelian groups which are naturally equipped with the structure of $\mathbb{Z}\left[x, x^{-1}\right]$-modules. In Section 3 we describe a relationship between congruences of a medial quandle that are below the orbit decomposition and submodules of the modules from which it is built. Then, in Section 4, we show that every finite SI medial quandle is either reductive, or affine and connected, and discuss briefly the connected case. In Section 5 we develop more structure theory of reductive medial quandles. Section 6 brings the main result, namely an explicit construction of reductive SI medial quandles (Theorem 6.12). In Section 7 we discuss with more details SI medial quandles with orbits which are cyclic groups. We also provide an example on an infinite non-connected and non-reductive medial quandle. We conclude with a few open questions related to the topic.

Notation and basic terminology. The identity permutation will always be denoted by 1. For two permutations $\alpha, \beta$, we write $\alpha^{\beta}=\beta \alpha \beta^{-1}$. The commutator is defined as $[\alpha, \beta]=\beta^{\alpha} \beta^{-1}$.

Let a group $G$ act on a set $X$. For $e \in X$, the stabilizer of $e$ will be denoted $G_{e}$.
Let $Q=(Q, \cdot)$ be a binary algebra. The left translation by $a \in Q$ is the mapping $L_{a}: Q \rightarrow Q$, $x \mapsto a x$. If $Q$ is a left quasigroup, the unique solution to $a u=b$ will be denoted by $u=a \backslash b$, and we have $L_{a}^{-1}(x)=a \backslash x$.

Notice that for an affine quandle $\operatorname{Aff}(A, f)$

$$
a \backslash b=L_{a}^{-1}(b)=\left(1-f^{-1}\right)(a)+f^{-1}(b) .
$$

Observe that $Q$ is left distributive iff all the left translations are endomorphisms, and $Q$ is a left quasigroup iff all the left translations are permutations. We will often use the following observation: for every $a \in Q$ and $\alpha \in \operatorname{Aut}(Q)$,

$$
\begin{equation*}
\left(L_{a}\right)^{\alpha}=L_{2} \tag{1.1}
\end{equation*}
$$

Occasionally, we will also use the right translations $R_{a}(x)=x a$. A quandle is called latin (or, a quasigroup), if the right translations, $R_{a}$, are bijective, too.

A subquandle is a subset closed with respect to both operations • and $\backslash$.
A quandle $(Q, \cdot)$ is (left) $m$-reductive, if it satisfies the identity

$$
(((x \underbrace{y) y) \ldots) y}_{m-\text { times }} \approx y .
$$

A quandle $Q$ is called reductive, if it is $m$-reductive for some $m$ and $Q$ is strictly $m$-reductive if it is $m$-reductive and not $k$-reductive for any $k<m$. In particular, $\operatorname{Aff}(A, 1)$ is strictly 1 -reductive and is called a (right) projection quandle.

## 2. Orbit decomposition

In this section, we recall notions and results from [7] on representing medial quandles as sums of affine meshes. We start with the definition of an important permutation groups acting on $Q$.

Definition 2.1. The (left) multiplication group of a quandle $Q$ is the permutation group generated by left translations, i.e.,

$$
\operatorname{LMlt}(Q)=\left\langle L_{a} \mid a \in Q\right\rangle \leq \operatorname{Aut}(Q)
$$

We define the displacement group as the subgroup

$$
\operatorname{Dis}(Q)=\left\langle L_{a} L_{b}^{-1} \mid a, b \in Q\right\rangle=\left\{L_{a_{1}}^{k_{1}} \ldots L_{a_{n}}^{k_{n}}: a_{1}, \ldots, a_{n} \in Q \text { and } \sum_{i=1}^{n} k_{i}=0\right\} .
$$

Both groups act naturally on $Q$ and it was proved in [4] that $\operatorname{LMlt}(Q)$ and $\operatorname{Dis}(\mathrm{Q})$ have the same orbits of action. We refer to the orbits of transitivity of the groups $\operatorname{LMlt}(Q)$ and $\operatorname{Dis}(Q)$ simply as the orbits of $Q$, and denote

$$
Q e=\{\alpha(e) \mid \alpha \in \operatorname{LMlt}(Q)\}=\{\alpha(e) \mid \alpha \in \operatorname{Dis}(Q)\}
$$

the orbit containing an element $e \in Q$. Notice that orbits are subquandles of $Q$.
One of the main results of [4] was that every orbit of a quandle $Q$ admits a certain group representation, called homogeneous representation, based on $\operatorname{Dis}(\mathrm{Q})$. In particular, if $Q$ has only one orbit (such a quandle is called connected) it has a Galkin's representation and the authors of [4] proved that the representation based on $\operatorname{Dis}(\mathrm{Q})$ is minimal such a representation.

This article deals with medial quandles. From the group-theoretical point of view, the importance of medial quandles comes from the fact that $\operatorname{Dis}(\mathrm{Q})$ is abelian.
Proposition 2.2 ([7]). Let $Q$ be a quandle. Then $Q$ is medial if and only if $\operatorname{Dis}(\mathrm{Q})$ is commutative.
As we said, every orbit of a medial quandle $Q$ admits a homogeneous representation based on $\operatorname{Dis}(\mathrm{Q})$ and the fact that $\operatorname{Dis}(\mathrm{Q})$ is abelian implies that the representation actually reduces to the definition of an affine quandle. The main result of [7] was a structural description of medial quandles based on their affine orbits - the tool we used to reconstruct the whole quandle from its affine pieces was the affine mesh.

Definition 2.3. An affine mesh over a non-empty set $I$ is a triple

$$
\mathcal{A}=\left(\left(A_{i}\right)_{i \in I} ;\left(\varphi_{i, j}\right)_{i, j \in I} ;\left(c_{i, j}\right)_{i, j \in I}\right)
$$

where $A_{i}$ are abelian groups, $\varphi_{i, j}: A_{i} \rightarrow A_{j}$ homomorphisms, and $c_{i, j} \in A_{j}$ constants, satisfying the following conditions for every $i, j, j^{\prime}, k \in I$ :
(M1) $1-\varphi_{i, i}$ is an automorphism of $A_{i}$;
(M2) $c_{i, i}=0$;
(M3) $\varphi_{j, k} \varphi_{i, j}=\varphi_{j^{\prime}, k} \varphi_{i, j^{\prime}}$, i.e., the following diagram commutes:

(M4) $\varphi_{j, k}\left(c_{i, j}\right)=\varphi_{k, k}\left(c_{i, k}-c_{j, k}\right)$.
If the index set is clear from the context, we shall write briefly $\mathcal{A}=\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$.
Definition 2.4. The sum of an affine mesh $\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$ over a set $I$ is a binary algebra defined on the disjoint union of the sets $A_{i}$, with operation

$$
a * b=c_{i, j}+\varphi_{i, j}(a)+\left(1-\varphi_{j, j}\right)(b)
$$

for every $a \in A_{i}$ and $b \in A_{j}$.
It was proved in [7] that the sum of any affine mesh is a medial quandle. Every fiber $A_{i}$ becomes a subquandle of the sum, and for $a, b \in A_{i}$ we have

$$
a * b=\varphi_{i, i}(a)+\left(1-\varphi_{i, i}\right)(b),
$$

hence $\left(A_{i}, *\right)$ is affine and equal to $\operatorname{Aff}\left(A_{i}, 1-\varphi_{i, i}\right)$. Moreover, every fiber turns out to be a union of orbits. If we want every fiber to be a single orbit, we have to add the indecomposability condition.

Definition 2.5. An affine mesh $\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$ over a set $I$ is called indecomposable if

$$
A_{j}=\left\langle\bigcup_{i \in I}\left(c_{i, j}+\operatorname{Im}\left(\varphi_{i, j}\right)\right)\right\rangle,
$$

for every $j \in I$. Equivalently, the group $A_{j}$ is generated by all the elements $c_{i, j}, \varphi_{i, j}(a)$ with $i \in I$ and $a \in A_{i}$.

Theorem 2.6. [7] A binary algebra is a medial quandle if and only if it is the sum of an indecomposable affine mesh. The orbits of the quandle coincide with the groups of the mesh.

Starting from a medial quandle $Q$, a natural way to define an indecomposable affine mesh that sums to $Q$ is the canonical mesh.

Definition 2.7. Let $Q$ be a medial quandle, and choose a transversal $E$ to the orbit decomposition. We define the canonical mesh for $Q$ over the transversal $E$ as $\mathcal{A}_{Q, E}=\left(\operatorname{Orb}_{Q}(e) ; \varphi_{e, f} ; c_{e, f}\right)$ with $e, f \in E$, where for every $x \in Q e$

$$
\varphi_{e, f}(x)=x f-e f \quad \text { and } \quad c_{e, f}=e f .
$$

Lemma 2.8. [7] Let $Q$ be a medial quandle and $\mathcal{A}_{Q, E}$ its canonical mesh. Then $\mathcal{A}_{Q, E}$ is an indecomposable affine mesh and $Q$ is equal to its sum.

Alternatively, we could have defined the canonical mesh using the groups $A_{e}=\operatorname{Dis}(Q) / \operatorname{Dis}(Q)_{e}$, homomorphisms $\varphi_{e, f}\left(\alpha \operatorname{Dis}(Q)_{e}\right)=\left[\alpha, L_{e}\right] \operatorname{Dis}(Q)_{f}$, and constants $c_{e, f}=L_{e} L_{f}^{-1} \operatorname{Dis}(Q)_{f}$. Then the original quandle $Q$ is isomorphic to the sum of the mesh, where the coset $\alpha \operatorname{Dis}(Q)_{e}$ corresponds to the element $\alpha(e) \in Q$.

## 3. Congruences below the orbit decomposition

As it was shown in Theorem 2.6, each medial quandle is the sum of an indecomposable affine mesh $\mathcal{A}=\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$ over the index set $I$ and the orbits of the quandle coincide with the groups of the mesh. All the abelian groups $A_{i}$ can be naturally equipped with the structure of a $\mathbb{Z}\left[x, x^{-1}\right]$-module by defining

$$
x^{n} \cdot a=\left(1-\varphi_{i, i}\right)^{n}(a), \quad \text { for all } n \in \mathbb{Z} \text { and } a \in A_{i} .
$$

Moreover, we have

$$
\varphi_{i, j}\left(x^{n} \cdot a\right)=\varphi_{i, j}\left(1-\varphi_{i, i}\right)^{n}(a)=\left(1-\varphi_{j, j}\right)^{n} \varphi_{i, j}(a)=x^{n} \varphi_{i, j}(a)
$$

and therefore every $\varphi_{i, j}$ can be treated as a $\mathbb{Z}\left[x, x^{-1}\right]$-module homomorphism. Hence, in the sequel, we shall assume that all the orbits are $R$-modules, where $R$ is a suitable image of $\mathbb{Z}\left[x, x^{-1}\right]$.

Let $Q$ be a medial quandle and $e \in Q$. Let $\alpha(e), \beta(e) \in Q e$ with $\alpha, \beta \in \operatorname{Dis}(Q)$ and put

$$
\alpha(e)+\beta(e)=\alpha \beta(e), \quad-\alpha(e)=\alpha^{-1}(e), \quad \text { and } \quad x^{n} \cdot \alpha(e)=\alpha^{L_{e}^{n}}(e)
$$

Then $\operatorname{Orb}_{Q}(e)=(Q e,+,-, e, \cdot)$ is a $\mathbb{Z}\left[x, x^{-1}\right]$-module, called the orbit module for $Q e$.
Let us note that the orbit decomposition provides a congruence, namely the relation $\pi \subseteq Q \times Q$ defined by

$$
a \pi b \text { iff } a=\alpha(b) \text { for some } \alpha \in \operatorname{Dis}(Q) .
$$

Clearly $\pi$ is an equivalence relation. Now let $a \pi b$ and $c \pi d$. Then $a=\alpha(b)$ and $c=\gamma(d)$ for some $\alpha, \gamma \in \operatorname{Dis}(Q)$. By commutativity of the group $\operatorname{Dis}(Q)$ we have

$$
\begin{array}{r}
a \cdot c=\alpha(b) \cdot \gamma(d)=L_{\alpha(b)} \gamma(d)=\alpha L_{b} \alpha^{-1} \gamma(d)=\alpha L_{b} \alpha^{-1} \gamma L_{d}^{-1}(d)= \\
L_{b} \alpha^{-1} \gamma L_{d}^{-1} \alpha(d)=L_{d} L_{d}^{-1} L_{b} \alpha^{-1} \gamma L_{d}^{-1} \alpha L_{d}(d)= \\
L_{d} \alpha^{-1} \gamma L_{d}^{-1} \alpha L_{d} L_{d}^{-1} L_{b}(d)=L_{d} \alpha^{-1} \gamma L_{d}^{-1} \alpha L_{b}(d)=L_{d} \alpha^{-1} \gamma L_{d}^{-1} \alpha(b \cdot d)
\end{array}
$$

which shows that $\pi$ is a quandle congruence.
Proposition 3.1. The relation $\pi$ is the least congruence on a quandle $Q$ such that the quotient $Q / \pi$ is the right projection quandle.

Proof. First note that, for any $a, b \in Q, a b=L_{a}(b)$, which means that $a b \pi b$ and shows that $Q / \pi$ is the right projection quandle.

Now, let $\psi$ be a congruence relation on $Q$ such that $Q / \psi$ is the right projection quandle. Then, for any $x, y \in Q, y \psi x y=L_{x}(y)$ and $y \psi(x \backslash y)=L_{x}^{-1}(y)$.

If $a \pi b$ then $a=\alpha(b)$, for some $\alpha \in \operatorname{Dis}(Q)$. By the definition of $\operatorname{Dis}(\mathrm{Q}), \alpha=L_{b_{1}}^{k_{1}} \ldots L_{b_{n}}^{k_{n}}$, for some $b_{1}, \ldots, b_{n} \in Q$, and $\sum_{i=1}^{n} k_{i}=0$. This gives the following:

$$
b \psi L_{b_{n}}^{k_{n}}(b) \psi L_{b_{n-1}}^{k_{n-1}} L_{b_{n}}^{k_{n}}(b) \psi \ldots \psi L_{b_{1}}^{k_{1}} \ldots L_{b_{n}}^{k_{n}}(b)=\alpha(b)=a
$$

So, $a \psi b$ and $\pi \subseteq \psi$.
Now we will describe a relationship between congruences of medial quandle $Q$ and congruences of the modules from which it is built. In particular, we will show that each congruence on $Q$, when restricted to an orbit, is a module congruence.

It is not usual to work with congruences of modules and we shall therefore be, from now on, speaking about submodules instead of congruences of modules. In particular, if $\varrho$ is a congruence of a module $M$ then there is a submodule $M_{\varrho}$ of $M$ such that $a \varrho b \Leftrightarrow a-b \in M_{\varrho}$. In the case if $a-b \in N$, for some submodule $N$, we will sometimes write $a \equiv_{N} b$.

Theorem 3.2. Let $Q$ be a medial quandle being the sum of an affine mesh ( $A_{i} ; \varphi_{i, j} ; c_{i, j}$ ) over a set $I$. Then $\varrho \subseteq \pi$ is a congruence relation on $Q$ if and only if, for each $i \in I$, there is a $\mathbb{Z}\left[x, x^{-1}\right]$-submodule $M_{i}$ of the module $A_{i}$ satisfying the condition

$$
\begin{equation*}
\varphi_{k, j}\left(M_{k}\right) \subseteq M_{j}, \text { for all } k, j \in I \tag{3.1}
\end{equation*}
$$

and such that for $a, b \in Q$

$$
\begin{equation*}
a \varrho b \text { iff } \exists(i \in I) a, b \in A_{i} \text { and } a \equiv_{M_{i}} b \tag{3.2}
\end{equation*}
$$

Proof. Let $\mathcal{A}_{Q, E}$ be the canonical mesh of $Q$ and $\varrho$ be a congruence relation on $Q$ such that $\varrho \subseteq \pi$. Let for some $x, y \in Q, x \varrho y$. Since $\varrho$ is a quandle congruence, it follows that for each $z \in Q$,

$$
z x=L_{z}(x) \varrho L_{z}(y)=z y
$$

and

$$
z \backslash x=L_{z}^{-1}(x) \varrho L_{z}^{-1}(y)=z \backslash y .
$$

In consequence, $\mu(x) \varrho \mu(y)$, for each $\mu \in \operatorname{Dis}(Q)$.
Let $a=\alpha(e), b=\beta(e), c=\gamma(e), d=\delta(e) \in Q e$, with $\alpha, \beta, \gamma, \delta \in \operatorname{Dis}(Q)$ and let $a \varrho b$ and $c \varrho d$. Thus, $\gamma(a) \varrho \gamma(b)$ and $\beta(c) \varrho \beta(d)$. Hence,

$$
c+a=\gamma(e)+\alpha(e)=\gamma \alpha(e)=\gamma(a) \varrho \gamma(b)=\gamma \beta(e)=\gamma(e)+\beta(e)=c+b,
$$

and similarly $(b+c) \varrho(b+d)$ which implies

$$
(a+c) \varrho(c+b) \varrho(b+d) .
$$

Therefore, $\varrho_{e}:=\left.\varrho\right|_{Q e}$ (the restriction of $\varrho$ to the orbit $Q e$ ) is a congruence relation of the abelian group $\operatorname{Orb}_{Q}(e)$.

Furthermore, it is easy to see that, for any $f \in E$,

$$
\varphi_{e, f}(a)=a f-e f \varrho_{f} b f-e f=\varphi_{e, f}(b) .
$$

In particular,

$$
x^{n} \cdot a=\left(1-\varphi_{e, e}\right)^{n}(a) \varrho_{e}\left(1-\varphi_{e, e}\right)^{n}(b)=x^{n} \cdot b
$$

and $\varrho_{e}$ is a congruence of the module $\operatorname{Orb}_{Q}(e)$. Then, for each $e \in E, M_{e}:=\left\{x \in Q e: x \varrho_{e} e\right\}$ is a submodule of the module $\operatorname{Orb}_{Q}(e)$. Hence, for each $x \in M_{e}$ and $f \in E$

$$
\varphi_{e, f}(x)=x f-e f \varrho_{f} e f-e f=f .
$$

So, conditions (3.1) and (3.2) are satisfied.
Now let $Q$ be the sum of an affine mesh $\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$ over a set $I$ and assume that there is, for each $i \in I$, a $\mathbb{Z}\left[x, x^{-1}\right]$-submodule $M_{i}$ of the module $A_{i}$ satisfying the condition (3.1). Let us define a relation $\varrho \subseteq Q \times Q$ in the following way

$$
a \varrho b \text { iff } \exists(i \in I) a, b \in A_{i} \text { and } a \equiv_{M_{i}} b .
$$

Let $a, b, c, d \in Q$ and suppose that $a \varrho b$ and $c \varrho d$. Then there are $i, j \in I$, such that $a, b \in A_{i}$, $a \equiv_{M_{i}} b, c, d \in A_{j}$ and $c \equiv_{M_{j}} d$. By Condition (3.1) we have $\varphi_{i, k}(a) \equiv_{M_{k}} \varphi_{i, k}(b)$ and $\varphi_{j, k}(c) \equiv_{M_{k}}$ $\varphi_{j, k}(d)$, for any $k \in I$.

By the definition of the multiplication in the sum of an affine mesh, we immediately obtain

$$
a c=c_{i, j}+\varphi_{i, j}(a)+\left(1-\varphi_{j, j}\right)(c) \equiv_{M_{j}} c_{i, j}+\varphi_{i, j}(b)+\left(1-\varphi_{j, j}\right)(d)=b d .
$$

Hence, $\varrho$ is a congruence of $Q$ such that $\varrho \subseteq \pi$.
Theorem 3.2 gives a one-to-one correspondence between congruences on $Q$ below $\pi$ and submodules of orbit modules satisfying the condition (3.1). Example 3.3 will show that just one submodule of any orbit module is sufficient to determine a congruence relation on $Q$ below $\pi$.

Example 3.3. Let $Q$ be a medial quandle which is the sum of an affine mesh $\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$ over a set $I$. Let, for some $i_{0} \in I, M_{i_{0}}$ be a $\mathbb{Z}\left[x, x^{-1}\right]$-submodule of $A_{i_{0}}$ and $M_{i}=\varphi_{i_{0}, i}\left(M_{i_{0}}\right)$, for $i \neq i_{0}$.

Note that $\varphi_{i_{0}, i_{0}}\left(M_{i_{0}}\right)=(1-x) \cdot M_{i_{0}} \subseteq M_{i_{0}}$ and, for $i \neq i_{0}, \varphi_{i, j}\left(M_{i}\right)=\varphi_{i, j} \varphi_{i_{0}, i}\left(M_{i_{0}}\right)=$ $\varphi_{j, j} \varphi_{i_{0}, j}\left(M_{i_{0}}\right)=\varphi_{j, j}\left(M_{j}\right)=(1-x) \cdot M_{j} \subseteq M_{j}$. Hence, the tuple of submodules $\left(M_{i}\right)_{i \in I}$ satisfies the condition (3.1).

Example 3.4. Let $Q$ be a medial quandle which is the sum of an affine mesh $\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$ over a set $I$ and assume that there exists $i_{0} \in I$ such that $\bigcap_{j \in I} \operatorname{Ker}\left(\varphi_{i_{0}, j}\right) \neq\{0\}$. Let $M_{i_{0}} \subseteq \bigcap_{j \in I} \operatorname{Ker}\left(\varphi_{i_{0}, j}\right)$ be a non-trivial $\mathbb{Z}\left[x, x^{-1}\right]$-submodule of $A_{i_{0}}$. Then, by Theorem 3.2, the relation

$$
a \alpha b \text { if and only if } a=b \text { or }\left(a, b \in A_{i_{0}} \text { and } a \equiv_{M_{i_{0}}} b\right)
$$

is a non-trivial congruence of $Q$, such that $\alpha \subseteq \pi$ and $\left.\alpha\right|_{A_{i}}$ is trivial, for each $i_{0} \neq i \in I$.

## 4. Subdirectly irreducible medial quandles

Let $m \geq 1$ be a natural number. Let $Q=\operatorname{Aff}(A, f)$ be an affine quandle. Then

$$
(((x \underbrace{y) y) \ldots) y}_{m-\text { times }}=(1-f)^{m}(x)+\left(1-(1-f)^{m}\right)(y),
$$

and $Q$ is $m$-reductive if and only if $(1-f)^{m}=0$. In particular, the orbit $A_{i}=\operatorname{Aff}\left(A_{i}, 1-\varphi_{i, i}\right)$ of a medial quandle is $m$-reductive if and only if $\varphi_{i, i}^{m}=0$. The following characterization of reductive medial quandles was presented in [7].
Theorem 4.1. [7] Let $Q$ be a medial quandle. Then the following statements are equivalent.
(1) $Q$ is reductive.
(2) At least one orbit of $Q$ is reductive.
(3) All the orbits of $Q$ are reductive.

Proposition 4.2. [7] Let $\mathcal{A}=\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$ be an indecomposable affine mesh over a set $I$. Then the sum of $\mathcal{A}$ is $m$-reductive if and only if, for every $i \in I$,

$$
\varphi_{i, i}^{m-1}=0
$$

In particular, a medial quandle $Q$ is $m$-reductive if and only if all the orbits of $Q$ are ( $m-1$ )reductive and is 2 -reductive if and only if every orbit is a projection quandle.

It was shown in [4] that a finite affine quandle is connected if and only if it is latin. Hence, by Proposition 4.2, it is clear that the only both connected and reductive finite medial quandle consists of exactly one element. In this Section we will prove that a non-connected finite SI medial quandle must be reductive.

Up to isomorphism there is only one two element SI medial quandle, namely the two element right projection quandle. Moreover, this is also the only 1-reductive SI medial quandle since each right projection quandle is the sum of one element orbits. In what follows we will study only SI medial quandles which have at least three elements.

Theorem 4.3. Let $Q$ be a finite subdirectly irreducible medial quandle such that $|Q|>2$. Then $Q$ is either connected (and therefore affine), or $Q$ is reductive.
Proof. Let $Q$ be a sum of an indecomposable affine mesh $\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$ over a set $I=\{1, \ldots, n\}$ for $n \geq 2$. Since $Q$ is finite, there exists $k_{i}$, for each $i \in I$, such that $\varphi_{i, i}^{k_{i}+1}\left(A_{i}\right)=\varphi_{i, i}^{k_{i}}\left(A_{i}\right)$. Let $k=\max \left\{k_{1}, \ldots, k_{n}\right\}$. Obviously, each $\varphi_{i, i}$ is a permutation on $\varphi_{i, i}^{k}\left(A_{i}\right)$, so $\operatorname{Ker} \varphi_{i, i} \cap \varphi_{i, i}^{k}\left(A_{i}\right)=\{0\}$. Let $i_{0}$ be an arbitrarily chosen element of the set $I$. Consider two congruence relations on the quandle $Q: \varrho_{1}$ given by $\operatorname{Ker} \varphi_{i_{0}, i_{0}}$ and $\varrho_{2}$ given by $\varphi_{i_{0}, i_{0}}^{k}\left(A_{i_{0}}\right)$, both constructed as in Example 3.3.

Let $B_{i}=\varphi_{i_{0}, i}\left(\operatorname{Ker} \varphi_{i_{0}, i_{0}}\right)$ and $C_{i}=\varphi_{i_{0}, i}\left(\varphi_{i_{0}, i_{0}}^{k}\left(A_{i_{0}}\right)\right)$, for $i \neq i_{0}$. Let $a \in B_{i}$. Then $a=\varphi_{i_{0}, i}(x)$ for some $x \in \operatorname{Ker} \varphi_{i_{0}, i_{0}}$ and $\varphi_{i, i}(a)=\varphi_{i, i} \varphi_{i_{0}, i}(x)=\varphi_{i_{0}, i} \varphi_{i_{0}, i_{0}}(x)=0$, so $B_{i} \subseteq \operatorname{Ker} \varphi_{i, i}$. On the other hand, $C_{i}=\varphi_{i_{0}, i}\left(\varphi_{i_{0}, i_{0}}^{k}\left(A_{i_{0}}\right)\right)=\varphi_{i, i}^{k}\left(\varphi_{i_{0}, i}\left(A_{i_{0}}\right)\right) \subseteq \varphi_{i, i}^{k}\left(A_{i}\right)$. One obtains that for each $i \neq i_{0}$, $B_{i} \cap C_{i}=\{0\}$ and $\varrho_{1} \cap \varrho_{2}$ is a trivial congruence on $Q$. Since $Q$ is subdirectly irreducible, there are two options:

- $\rho_{2}$ is trivial and hence $\varphi_{i, i}^{k}=0$, for each $i$. By Theorem 4.1 this means that $Q$ is reductive;
- $\rho_{1}$ is trivial and hence $\varphi_{i, i}$ is a bijection, for each $i$. By results of [7, Section 5] this means that $Q$ is a product of an affine quandle and a projection quandle. $Q$ is subdirectly irreducible and therefore $Q$ is either a projection quandle (and hence reductive) or an affine quandle (and hence connected).

Theorem 4.3 cannot be extended to infinite quandles because there exist infinite SI medial quandles that are neither reductive nor connected, see e.g. Theorem 7.4. On the other hand, a SI medial quandle with at least three elements cannot be both connected and reductive or nonconnected and affine.

Proposition 4.4. A connected non-trivial medial quandle $Q$ is never reductive.
Proof. A connected medial quandle has only one orbit. Hence the mapping $\varphi_{1,1}$ in its canonical mesh has to be surjective. And therefore $\varphi_{1,1}^{n}=0$ for no number $n$.

Proposition 4.5. The only affine non-connected subdirectly irreducible medial quandle has two elements.

Proof. Suppose $Q=\operatorname{Aff}(A, f)$. The only SI projective quandle has two elements. Consider thus that $Q$ is not projective, i.e. that $\pi$ is non-trivial. All the orbits of $Q$ are cosets of $(1-f)(A)$. There are now two possibilities:

- If $\operatorname{Im}(1-f) \cap \operatorname{Ker}(1-f)=\{0\}$ then the congruence $\pi \cap \equiv_{\operatorname{Ker}(1-f)}$ is trivial.
- If $\operatorname{Im}(1-f) \cap \operatorname{Ker}(1-f)$ is non-trivial then let $a \notin \operatorname{Im}(1-f)$ and consider $M_{1}=\operatorname{Im}(1-f) \cap$ $\operatorname{Ker}(1-f)$ and $M_{2}=M_{1}+a$. Let us define $\equiv_{M_{i}}$, for $i \in\{1,2\}$, as

$$
a \equiv_{M_{i}} b \quad \text { if } a=b \text { or } a-b \in M_{i} .
$$

It is easy to see that both the relations $\equiv_{M_{i}}$ are congruences. Since $M_{1}$ and $M_{2}$ are disjoint, clearly $\equiv_{M_{1}} \cap \equiv_{M_{2}}$ is trivial.

Consider now a connected affine quandle $Q=\operatorname{Aff}(A, f)$. By Theorem 3.2, congruences of $Q$ and congruences of the $\mathbb{Z}\left[x, x^{-1}\right]$-module $A$ coincide. In particular, if $Q$ is SI, the smallest non-trivial congruence of $Q$ is equal to the smallest non-trivial submodule of $A$. Consequently, the quandle $Q$ is subdirectly irreducible if and only if $A$ is a subdirectly irreducible $\mathbb{Z}\left[x, x^{-1}\right]$-module.

In the previous paragraph, one direction did not need the assumption of connectedness, while the other needed. Indeed, SI modules can give non-connected quandles that are not SI.

Example 4.6. Consider the affine quandle $A f f\left(\mathbb{Z}_{4}, 3\right)$. It is a sum of the affine mesh $\left(\left(\mathbb{Z}_{2}, \mathbb{Z}_{2}\right) ;\left(\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right) ;\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)\right)$. It is clear that there are two minimal elements in the lattice of its congruences: $\{\{0,2\},\{1\},\{3\}\}$ and $\{\{1,3\},\{0\},\{2\}\}$, so $\operatorname{Aff}\left(\mathbb{Z}_{4}, 3\right)$ is not subdirectly irreducible. However, $\mathbb{Z}_{4}=Z_{2^{2}}$ is a subdirectly irreducible $\mathbb{Z}$-module.

Recall that using Kearnes's classification of SI modes [6] one can conclude that all SI quandles are either quasi-affine or of the set type. Since affine quandles are trivially quasi-affine, Theorem 4.3 shows that all finite SI reductive medial quandles are of the set type.

## 5. Reductivity versus nilpotency

Reductivity of medial quandles is closely related to the notion of nilpotency: reductive affine quandles are reducts of modules over nilpotent rings, namely rings $\mathbb{Z}\left[x, x^{-1}\right] /(1-x)^{m}$, for some $m \geq 1$. A medial quandle $Q$ is reductive if and only if its left multiplication group $\operatorname{LMlt}(Q)$ is nilpotent, as we shall see later in Theorem 5.3. Further, a nilpotent $\varphi_{i, i}$ can appear in an indecomposable affine mesh $\mathcal{A}=\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$ if and only if the sum of $\mathcal{A}$ is reductive.

In the sequel we focus on the nilpotency of $\operatorname{LMlt}(Q)$. We start with two auxiliary lemmas.
Lemma 5.1. Let $Q$ be a medial quandle, $e, f \in Q$ and $\alpha \in \operatorname{Dis}(\mathrm{Q})$. Then $\alpha^{L_{e}}=\alpha^{L_{f}}$.
Proof. $\alpha^{L_{e}}=L_{e} \alpha L_{e}^{-1}=L_{f} L_{f}^{-1} L_{e} \alpha L_{e}^{-1}=L_{f} \alpha L_{f}^{-1} L_{e} L_{e}^{-1}=\alpha^{L_{f}}$.
By $G^{\prime}$ we will denote the commutator subgroup (or derived subgroup) of a group $G$, i.e. the subgroup generated by all the commutators of the group $G$.

It was proved in [4] that for any quandle $Q, \operatorname{LMlt}(Q)^{\prime} \unlhd \operatorname{Dis}(\mathrm{Q})$. If $Q$ is medial, it means that $\operatorname{LMlt}(Q)^{\prime}$ is abelian and therefore $\left[\operatorname{LMlt}(Q)^{\prime}, \operatorname{LMlt}(Q)^{\prime}\right]=1$. Although the following lemma is pronounced for general groups, its setting in our context results in a strong relation to nilpotency.
Lemma 5.2. Let $G$ be a group with $\left[G^{\prime}, G^{\prime}\right]=1$. Then, for each $\alpha, \beta \in G^{\prime}$ and $a, b, c, d \in G$ :
(1) $[a, b c]=[a, b] \cdot[a, c]^{b}$,
(2) $[\alpha \beta, c]=[\alpha, c] \cdot[\beta, c]$,
(3) $\left[\alpha^{b}, c\right]=[\alpha, c]^{b^{c}}$,
(4) $[[a, b c], d]=[[a, b], d] \cdot[[a, c], d]^{d^{d}}$.

Proof. (1) $[a, b c]=a b c a^{-1} c^{-1} b^{-1}=a b a^{-1}[a, c] b^{-1}=[a, b] b[a, c] b^{-1}=[a, b] \cdot[a, c]^{b}$
(2) $[\alpha \beta, c]=\alpha \beta c \beta^{-1} \alpha^{-1} c^{-1}=\alpha[\beta, c] c \alpha^{-1} c^{-1}=\alpha c \alpha^{-1} c^{-1}[\beta, c]=[\alpha, c] \cdot[\beta, c]$
(3) $\left[\alpha^{b}, c\right]=b \alpha b^{-1} c b \alpha^{-1} b^{-1} c^{-1}=b \alpha\left[b^{-1}, c\right] c \alpha^{-1} b^{-1} c^{-1}=b\left[b^{-1}, c\right] \alpha c \alpha^{-1} b^{-1} c^{-1}=b^{c}[\alpha, c] b^{-1 c}=$ $[\alpha, c]^{b^{c}}$
(4) $[[a, b c], d]=\left[[a, b] \cdot[a, c]^{b}, d\right]=[[a, b], d] \cdot\left[[a, c]^{b}, d\right]=[[a, b], d] \cdot[[a, c], d]^{b^{d}}$

Theorem 5.3. Let $Q$ be a medial quandle and let $m \geq 1$. Then $Q$ is strictly m-reductive if and only if $\operatorname{LMlt}(Q)$ is nilpotent of degree $m-1$.

Proof. A quandle is a projection quandle if and only if its left multiplication group is trivial, so theorem holds for $m=1$.

Let $m>1$ and let $\mathcal{A}=\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$, for $i, j \in I$, be the canonical affine mesh for $Q$. Choosing an element $e_{i} \in A_{i}$, every element of $A_{i}$ can be written as $\alpha\left(e_{i}\right)$, for some $\alpha \in \operatorname{LMlt}(Q)$. Moreover, according to [7, Lemma 3.8], we have $\varphi_{i, j}\left(\alpha\left(e_{i}\right)\right)=\left[\alpha, L_{e_{i}}\right]\left(e_{j}\right)$, for any $\alpha \in \operatorname{LMlt}(Q)$ and $i, j \in I$.

According to Proposition 4.2, $Q$ is $m$-reductive if and only if $\varphi_{i, i}^{m-1}=0$, for each $i \in I$. This condition can be equivalently rewritten as $\left[\ldots\left[\left[\alpha, L_{e_{i}}\right], L_{e_{i}}\right], \ldots, L_{e_{i}}\right]\left(e_{i}\right)=e_{i}$, for each $i \in I$ and $\alpha \in \operatorname{LMlt}(Q)$, which means $\left[\ldots\left[\left[\alpha, L_{e_{i}}\right], L_{e_{i}}\right], \ldots, L_{e_{i}}\right]=1$.
" $\Leftarrow$ " If $\operatorname{LMlt}(Q)$ is nilpotent of degree $m-1$ then $\left[\ldots\left[\alpha_{1}, \alpha_{2}\right], \ldots, \alpha_{m}\right]=1$, for any $\alpha_{j} \in \operatorname{LMlt}(Q)$, in particular for $\alpha_{2}=\cdots=\alpha_{m}=L_{e_{i}}$, for any $i \in I$.
" $\Rightarrow$ " According to Lemma 5.1, we have $\left[\left[\alpha, L_{e_{i}}\right], L_{e_{i}}\right]=\left[\left[\alpha, L_{e_{i}}\right], L_{e_{j}}\right]$, for any $i, j$. Hence $m$ reductivity implies $\left[\ldots\left[\left[\alpha, L_{e_{i_{1}}}\right], L_{e_{i_{2}}}\right], \ldots, L_{e_{i_{m-1}}}\right]=1$, for any $e_{i_{k}}, 1 \leq k \leq m-1$. Now we should inductively enlarge this property to $\left[\ldots\left[\left[\alpha, \beta_{1}\right], \beta_{2}\right], \ldots, \beta_{m-1}\right]$, for any $\beta_{1}, \ldots, \beta_{m-1} \in \operatorname{LMlt}(Q)$. Suppose $\beta_{j}=\hat{\beta}_{j} \bar{\beta}_{j}$. According to Lemma 5.2 (4),

$$
\begin{aligned}
& {\left[\ldots\left[\ldots\left[\left[\alpha, \beta_{1}\right], \beta_{2}\right], \ldots, \hat{\beta}_{j} \bar{\beta}_{j}\right], \ldots, \beta_{m-1}\right]=} \\
& \quad\left[\ldots\left[\ldots\left[\left[\alpha, \beta_{1}\right], \beta_{2}\right], \ldots, \hat{\beta}_{j}\right], \ldots, \beta_{m-1}\right] \cdot\left[\ldots\left[\ldots\left[\left[\alpha, \beta_{1}\right], \beta_{2}\right], \ldots, \bar{\beta}_{j}\right], \ldots, \beta_{m-1}\right]^{\hat{\beta}_{j}^{\beta_{j+1}}}{ }^{\beta_{m-1}}
\end{aligned}
$$

and this is trivial, due to the induction hypothesis. Hence $\operatorname{LMlt}(Q)$ is a nilpotent group of degree at most $m-1$.

By Proposition 4.2 we know that the orbits of an $m$-reductive medial quandle are ( $m-1$ )reductive. But not necessarily strictly $(m-1)$-reductive - the degree of reductivity may be even smaller. It was nevertheless proved in [7] that the orbits cannot be ( $m-4$ )-reductive in such a case. We improve the result here showing that neither $(m-3)$-reductive orbits can appear.

Lemma 5.4. Let $m \geq 1$ and $\mathcal{A}=\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$ be an indecomposable affine mesh over a set $I$. Assume there is $j \in I$ such that $\varphi_{j, j}^{m}=0$. Then $\varphi_{i, i}^{m+1}=0$ for every $i \in I$.
Proof. First note that applying (M3) $m$-times, for any $k \in I$, we have

$$
\begin{equation*}
\varphi_{k, k}^{m} \varphi_{j, k}=\varphi_{j, k} \varphi_{j, j}^{m}=0, \tag{5.1}
\end{equation*}
$$

because $\varphi_{j, j}^{m}=0$ by assumption.
The indecomposability condition says that the group $A_{k}$ is generated by all the elements $c_{i, k}$, $\varphi_{i, k}(a)$ with $i \in I$ and $a \in A_{i}$. So it is sufficient to verify that $\varphi_{k, k}^{m+1} \varphi_{i, k}=0$ and $\varphi_{k, k}^{m+1}\left(c_{i, k}\right)=0$, for every $i \in I$.

By (5.1) and (M4) we have

$$
\varphi_{k, k}^{m+1}\left(c_{i, k}-c_{j, k}\right)=\varphi_{k, k}^{m} \varphi_{j, k}\left(c_{i, j}\right)=0 .
$$

This implies

$$
\varphi_{k, k}^{m+1}\left(c_{i, k}\right)=\varphi_{k, k}^{m+1}\left(c_{j, k}\right),
$$

for all $i, k \in I$. In particular, for $i=k$, we see that $\varphi_{k, k}^{m+1}\left(c_{k, k}\right)=0$, and thus $\varphi_{k, k}^{m+1}\left(c_{j, k}\right)=0$. This gives $\varphi_{k, k}^{m+1}\left(c_{i, k}\right)=0$ for every $i \in I$.

Further, by (M3) applying $(m+1)$-times,

$$
\varphi_{k, k}^{m+1} \varphi_{i, k}=\varphi_{j, k} \varphi_{j, j}^{m} \varphi_{i, j}=0
$$

for every $i \in I$. Hence, $\varphi_{k, k}^{m+1}=0$, for every $k \in I$.
Corollary 5.5. Let $Q$ be a medial quandle. If one orbit of $Q$ is $m$-reductive, then $Q$ is $(m+2)$ reductive.

We recall now a few results about 2-reductive medial quandles.
Lemma 5.6. [7] Let $\mathcal{A}=\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$ be an indecomposable affine mesh over a set I. Assume there are $j, k \in I$ such that $\varphi_{j, k}=0$. Then $\varphi_{i, k}=0$ for every $i \in I$.
Theorem 5.7. [7] Let $Q$ be a medial quandle and assume it is the sum of an indecomposable affine mesh $\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$ over a set $I$. Then the following statements are equivalent.
(1) $Q$ is 2-reductive.
(2) For every $j \in I$, there is $i \in I$ such that $\varphi_{i, j}=0$.
(3) $\varphi_{i, j}=0$ for every $i, j \in I$.

In particular, medial quandles with a one-element orbit are always 2-reductive and with a twoelement orbit are 3 -reductive.

We know by now that a strictly $m$-reductive medial quandle has $(m-1)$-reductive orbits and may have $(m-2)$-reductive orbits too. Some of the orbits might even be isomorphic. But none of the mappings $\varphi_{i, j}$ is a permutation.

Proposition 5.8. Let $Q$ be a reductive medial quandle which is the sum of an indecomposable affine mesh $\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$ over a set $I$. If, for $i, k \in I,\left|A_{i}\right|>1$ or $\left|A_{k}\right|>1$, then the homomorphism $\varphi_{i, k}$ is not a permutation.

Proof. Since medial quandles with a one-element orbit are always 2-reductive and by Theorem 5.7, in 2-reductive medial quandles $\varphi_{l, j}=0$, for every $l, j \in I$, we may assume that $\left|A_{i}\right|>1$.

Let us suppose that there are $i, k \in I$ such that $\varphi_{i, k}: A_{i} \rightarrow A_{k}$ is a permutation. By (M3) $\varphi_{k, k} \varphi_{i, k}=\varphi_{i, k} \varphi_{i, i}$. It implies

$$
\begin{equation*}
\varphi_{i, k}^{-1} \varphi_{k, k}=\varphi_{i, i} \varphi_{i, k}^{-1} . \tag{5.2}
\end{equation*}
$$

On the other hand, by (M4) we have

$$
c_{j, i}=\varphi_{i, k}^{-1} \varphi_{k, k}\left(c_{j, k}-c_{i, k}\right)=\varphi_{i, i} \varphi_{i, k}^{-1}\left(c_{j, k}-c_{i, k}\right) \in \operatorname{Im}\left(\varphi_{i, i}\right),
$$

for any $j \in I$. By indecomposability, it means that $A_{i}=\left\langle\operatorname{Im}\left(\varphi_{j, i}\right): j \in I\right\rangle$.
Since $Q$ is a reductive medial quandle, there is a natural number $m>0$ such that $\varphi_{k, k}^{m}=0$. Let $m$ be the least such a number. If $m=1$, then by Lemma $5.6, \varphi_{j, k}=0$, for every $j \in I$. In particular, $\varphi_{i, k}=0$.

For $m>1$, once again by (M3) one has

$$
\begin{equation*}
\varphi_{k, k}^{m-1} \varphi_{i, k} \varphi_{j, i}=\varphi_{k, k}^{m-1} \varphi_{k, k} \varphi_{j, k}=\varphi_{k, k}^{m} \varphi_{j, k}=0, \tag{5.3}
\end{equation*}
$$

for any $j \in I$. Since $\varphi_{i, k}$ is a permutation and $A_{i}=\left\langle\operatorname{Im}\left(\varphi_{j, i}\right): j \in I\right\rangle$, Condition (5.3) implies that $\varphi_{k, k}^{m-1}=0$, a contradiction with the minimality of $m$. Hence there are not $i, k \in I$ such that $\varphi_{i, k}$ is a permutation.

At the end of this section we shall prove that reductive medial quandles have a property enabling us to find small congruences that are good candidates for monoliths.

Lemma 5.9. Let $m \geq 2$ and $Q$ be an indecomposable affine mesh $\mathcal{A}=\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$ over a set $I$ and assume that for some $i, j \in I, i \neq j, \varphi_{i, i}^{m-1}=\varphi_{j, j}^{m-1}=0$. Then $\varphi_{i, j} \varphi_{i, i}^{m-2}=\varphi_{j, j}^{m-2} \varphi_{i, j}=0$.
Proof. For $m=2$ the conclusion follows by Lemma 5.6. Let $m>2$. Clearly, by (M3),

$$
\begin{equation*}
\varphi_{i, j} \varphi_{i, i}^{m-2}=\varphi_{i, j} \varphi_{i, i}^{m-3} \varphi_{i, i}=\varphi_{j, j} \varphi_{j, j}^{m-3} \varphi_{i, j}=\varphi_{j, j}^{m-2} \varphi_{i, j} \tag{5.4}
\end{equation*}
$$

By the indecomposability it is sufficient to verify that $\varphi_{i, j} \varphi_{i, i}^{m-2} \varphi_{k, i}=0$ and $\varphi_{i, j} \varphi_{i, i}^{m-2}\left(c_{k, i}\right)=0$, for every $k \in I$.

For each $k \in I$, by (M3) we have

$$
\varphi_{i, j} \varphi_{i, i}^{m-2} \varphi_{k, i}=\varphi_{i, j} \varphi_{i, i} \varphi_{i, i}^{m-3} \varphi_{k, i}=\varphi_{j, j} \varphi_{j, j} \varphi_{j, j}^{m-3} \varphi_{k, j}=\varphi_{j, j}^{m-1} \varphi_{k, j}=0
$$

and by (5.4) and (M4)

$$
\varphi_{i, j} \varphi_{i, i}^{m-2}\left(c_{k, i}\right)=\varphi_{j, j}^{m-2} \varphi_{i, j}\left(c_{k, i}\right)=\varphi_{j, j}^{m-2} \varphi_{j, j}\left(c_{k, j}-c_{i, j}\right)=\varphi_{j, j}^{m-1}\left(c_{k, j}-c_{i, j}\right)=0
$$

Proposition 5.10. Let $Q$ be a non-projective reductive quandle that is the sum of an affine mesh $\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$ over a set $I$. Then there exists $i \in I$ such that $\bigcap_{j \in I} \operatorname{Ker}\left(\varphi_{i, j}\right) \neq\{0\}$.

Proof. Assume that $Q$ is strictly $m$-reductive, for some $m \geq 2$. If $m=2$ then by Theorem 5.7, $\varphi_{i, j}=0$, for all $i, j \in I$.

Suppose now $m>2$. Since $Q$ is strictly $m$-reductive then by Proposition 4.2, there is at least one orbit of $Q$, say $A_{1}$, which is strictly $(m-1)$-reductive. Hence, there is an element $0 \neq a_{1} \in A_{1}$ such that the elements: $0, a_{1}, \varphi_{1,1}\left(a_{1}\right), \ldots, \varphi_{1,1}^{m-2}\left(a_{1}\right) \in A_{1}$ are pairwise different. By Lemma 5.9, $0 \neq \varphi_{1,1}^{m-2}\left(a_{1}\right) \in \bigcap_{j \in I} \operatorname{Ker}\left(\varphi_{1, j}\right)$.

As we have seen in the proof, every strictly $(m-1)$-reductive orbit has the property hence if $\bigcap_{j \in I} \operatorname{Ker}\left(\varphi_{i, j}\right)=\{0\}$, for some $i \in I$, then the orbit has to be $(m-2)$-reductive.

## 6. Subdirectil irreducible non-Connected quasi-Reductive quandles

The aim of this section is to describe all finite subdirectly irreducible $m$-reductive quandles, for $m \geq 2$. As we have already noticed in Section 4 , such reductive quandles are always non-connected. Nevertheless, it turns out that all the considerations in this section can be performed for a slightly larger class of quandles, let us call them quasi-reductive, and we shall thus obtain some infinite examples that are not necessarily reductive.

We start the section with the construction of a congruence that is going to be the monolith of our examples; the nature of this congruence will justify our definition of quasi-reductivity.

Let $Q$ be a quandle and let $\lambda$ be the relation on $Q$ defined by

$$
a \lambda b \text { iff } \forall(x \in Q) a x=b x \text {. }
$$

Obviously, $\lambda$ is a congruence on $Q$ (see [12]). Set

$$
\begin{equation*}
\theta:=\pi \cap \lambda . \tag{6.1}
\end{equation*}
$$

Let a medial quandle $Q$ be the sum of an indecomposable affine mesh $\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$ over a set $I$. Note that for each $i \in I, M_{\left.\theta\right|_{A_{i}}}=\bigcap_{j \in I} \operatorname{Ker}\left(\varphi_{i, j}\right)$. To see it, let $a, b \in A_{i}$. Then

$$
\begin{array}{r}
a \theta b \Leftrightarrow \forall(j \in I) \forall\left(x \in A_{j}\right) a x=b x \Leftrightarrow \\
\forall(j \in I) \forall\left(x \in A_{j}\right) c_{i, j}+\varphi_{i, j}(a)+\left(1-\varphi_{j, j}\right)(x)=c_{i, j}+\varphi_{i, j}(b)+\left(1-\varphi_{j, j}\right)(x) \Leftrightarrow \\
\forall(j \in I) \varphi_{i, j}(a)=\varphi_{i, j}(b) \Leftrightarrow \forall(j \in I) a-b \in \operatorname{Ker}\left(\varphi_{i, j}\right) \Leftrightarrow a-b \in \bigcap_{j \in I} \operatorname{Ker}\left(\varphi_{i, j}\right) .
\end{array}
$$

Definition 6.1. Let $Q$ be a medial quandle being the sum of an indecomposable affine mesh $\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$ over a set $I$. The quandle is called quasi-reductive if there exists $i \in I$ such that $\bigcap_{j \in I} \operatorname{Ker}\left(\varphi_{i, j}\right)$ is non-trivial.

An alternative definition of quasi-reductivity says that a medial quandle is quasi-reductive if and only if $\theta$ is non-trivial on $Q$. The class of quasi-reductive medial quandles contains all non-projective reductive medial modes, according to Proposition 5.10. But not every quasi-reductive is reductive, an example is $\operatorname{Aff}\left(\mathbb{Z}_{6},-1\right)$. Moreover, there exist even SI affine quasi-reductive medial quandles.

Example 6.2. Take $Q=\operatorname{Aff}\left(\mathbb{Z}_{p^{\infty},}, 1-p\right)$, where $p$ is a prime and $\mathbb{Z}_{p \infty}$ is the Prüfer group. The multiplication by $p$ is surjective on $\mathbb{Z}_{p \infty}$ and therefore $Q$ is connected. The multiplication by $p$ is not injective, its kernel is $\left\{a / p ; a \in \mathbb{Z}_{p}\right\}$, i.e. the socle of $\mathbb{Z}_{p^{\infty}}$, and therefore $Q$ is quasi-reductive. This socle is the minimal $\mathbb{Z}$-submodule of $\mathbb{Z}_{p^{\infty}}$ and corresponds to the monolith of the subdirectly irreducible quandle $Q$.

As we said, the main goal of the section is to describe all finite reductive SI medial quandles and hence, from now on, we shall suppose not only that $Q$ is quasi-reductive but also non-connected. Moreover, in the next sequence of lemmas we assume that $Q$ is a subdirectly irreducible and that $Q$ is the sum of an indecomposable affine mesh $\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$ over at least two element set $I$. We also assume that the orbit $A_{1}$ is an orbit where $\theta$ is non-trivial, i.e. that $\bigcap_{j \in I} \operatorname{Ker}\left(\varphi_{1, j}\right) \neq\{0\}$. If $Q$ happens to be reductive, we could alternatively say that $A_{1}$ is strictly ( $m-1$ )-reductive. We show first that $A_{1}$ is the only orbit where $\theta$ is non-trivial (or that is not ( $m-2$ )-reductive).
Lemma 6.3. Let $1 \neq i \in I$ and $a, b \in A_{i}$. Then a $\theta b$ if and only if $a=b$.
Proof. Let $M_{1}=\bigcap_{j \in I} \operatorname{Ker}\left(\varphi_{1, j}\right)$ and consider submodules $M_{j}=\varphi_{1, j}\left(M_{1}\right)$, for $j>1$. By Example 3.3, there is a congruence $\varrho \subseteq \pi$ of $Q$ such that, for each $k \in I, M_{\left.\varrho\right|_{A_{k}}}=M_{k}$. Note that $M_{k}$ is trivial, for $k>1$.

Suppose, on the contrary, that there are $1 \neq i \in I$ and elements $a \neq b \in A_{i}$ such that $a \theta b$. Let $M_{i}^{\prime}=\bigcap_{j \in I} \operatorname{Ker}\left(\varphi_{i, j}\right) \neq\{0\}$ and consider submodules $M_{j}^{\prime}=\varphi_{i, j}\left(M_{i}^{\prime}\right)$, for $i \neq j$. By Example 3.3 again, there is a congruence $\varrho^{\prime} \subseteq \pi$ of $Q$ such that, for each $k \in I, M_{\left.\varrho^{\prime}\right|_{A_{k}}}=M_{k}^{\prime}$. Again, $M_{k}^{\prime}$ is trivial, for $k \neq j$.

It follows that $\varrho \cap \varrho^{\prime}=\Delta$, which contradicts the assumption that $Q$ is subdirectly irreducible.
Lemma 6.4. The $R$-module $A_{1}$ is subdirectly irreducible.
Proof. Suppose, contrary to our claim, that there are two distinct minimal $R$-submodules of $A_{1}$, let us say $S_{1}$ and $S_{2}$. It follows from Example 3.3 that there are congruences $\xi_{1}$ and $\xi_{2}$ of the quandle $Q$, such that $M_{\left.\xi_{1}\right|_{A_{1}}}=S_{1}$ and $M_{\left.\xi_{2}\right|_{A_{1}}}=S_{2}$. By Theorem 3.2, $M=M_{\left.\theta\right|_{A_{1}}}$ is a non-trivial $R$-submodule of $A_{1}$. Let us consider two cases:

Case 1. Let $S_{1} \cap M=\{0\}$. By assumption, $M_{\theta \cap \xi_{1} \mid A_{1}}=M \cap S_{1}=\{0\}$. Moreover, by Lemma 6.3, we obtain $\theta \cap \xi_{1}=\Delta$. Similarly we proceed in the case $S_{2} \cap M=\{0\}$.

Case 2. Assume now $S_{1} \subset M$ and $S_{2} \subset M$. Then congruences $\xi_{1}$ and $\xi_{2}$ are non-trivial congruences of the quandle $Q$, with $\xi_{1} \subset \theta$ and $\xi_{2} \subset \theta$. Therefore $\xi_{1} \cap \xi_{2}=\Delta$ due to Lemma 6.3.

In both cases, $Q$ is not subdirectly irreducible, contrary to the assumption.
Let $a, b \in Q$ and $\Theta(a, b)$ denote the smallest congruence on $Q$ collapsing $(a, b)$. Recall, by [10, Theorem 1.20.] $\Theta(a, b)$ can be described by the following recursion:

$$
\begin{aligned}
X_{0} & =\{(a, b),(b, a)\} \cup\{(q, q): q \in Q\} \\
X_{n+1} & =X_{n} \cup\left\{\left(x \cdot x^{\prime}, y \cdot y^{\prime}\right):(x, y),\left(x^{\prime}, y^{\prime}\right) \in X_{n}\right\} \cup\left\{(x, z):(x, y),(y, z) \in X_{n} \text { for some } y \in Q\right\} .
\end{aligned}
$$

Then $\Theta(a, b)=\bigcup_{n \in \mathbb{N}} X_{n}$.
Lemma 6.5. Let $a, b \in Q \backslash A_{1}$. If for each $x \in A_{1}, a x=b x$, then $a=b$.
Proof. Let $a \neq b \in Q \backslash A_{1}$ and $a x=b x$, for every $x \in A_{1}$. Then $\left.\Theta(a, b)\right|_{A_{1}}=\Delta$. By Lemma 6.3, the relation $\left.\theta\right|_{\left(Q \backslash A_{1}\right)}$ is also trivial. It follows that $\Theta(a, b) \cap \theta=\Delta$, so $Q$ can not be subdirectly irreducible.

Corollary 6.6. For each $1 \neq i \in I, \varphi_{i, 1}\left(A_{i}\right)$ embeds into the $R$-module $A_{1}$.
Proof. We show that for each $1 \neq i \in I, \varphi_{i, 1}$ is an injection. Let $\varphi_{i, 1}(a)=\varphi_{i, 1}(b)$ for some $a, b \in A_{i}$. Hence, for any $x \in A_{1}$

$$
a \cdot x=c_{i, 1}+\varphi_{i, 1}(a)+\left(1-\varphi_{1,1}\right)(x)=c_{i, 1}+\varphi_{i, 1}(b)+\left(1-\varphi_{1,1}\right)(x)=b \cdot x .
$$

Hence, by Lemma 6.5, $a=b$.
Summarizing, by Lemmas 6.3 and 6.4, Corollaries 5.5 and 6.6 in the reductive case, we have that a SI strictly $m$-reductive medial quandle has exactly one strictly $(m-1)$-reductive orbit $A_{1}$, which is a subdirectly irreducible $R$-module. Furthermore, for each $1 \neq i \in I, A_{i}$ is strictly $(m-2)$-reductive quandle and for every $1 \neq i \in I$, the homomorphisms $\varphi_{i, 1}$ are injective.

By Corollary 6.6 , we can assume that, for each $1 \neq i \in I$, the orbit $A_{i}$ is an $R$-submodule of $A_{1}$, $\varphi_{i, i}=\varphi_{1,1}$, and $\varphi_{i, 1}=1_{A_{i}}$. Furthermore, by (M3) it follows that for each $1<i, j \in I$

$$
\begin{gathered}
\varphi_{1, i}=\varphi_{1, i} \varphi_{i, 1}=\varphi_{i, i}^{2}=\varphi_{1,1}^{2}, \\
\varphi_{i, j}=\varphi_{j, 1} \varphi_{i, j}=\varphi_{1,1} \varphi_{i, 1}=\varphi_{1,1} .
\end{gathered}
$$

Each fiber $A_{i}$ can be structurally viewed either as an $R$-module or as a permutation group acting on $Q$. We need both the features and therefore the fibres will be treated either as modules or as permutation groups, according to our needs.

Let, for each $j \in I$, denote by $0_{j}$ the neutral element of $A_{j}$.
Lemma 6.7. For each $1<i \neq j \in I, c_{i, 1} \neq c_{j, 1}$.

Proof. Suppose, that there are $1<i \neq j \in I$ such that $c_{i, 1}=c_{j, 1}$. Then for all $x \in A_{1}, 0_{i} \cdot x=0_{j} \cdot x$. Hence, by Lemma 6.5, $0_{i}=0_{j}$, a contradiction.

Lemma 6.8. For each $1<i \neq j \in I$, the constants $c_{i, j}$ are uniquely determined only by the constants $c_{i, 1} \in A_{1}$.

Proof. It straightforwardly follows by (M4) that, for any $i, j, k \in I, \varphi_{j, k}\left(c_{i, j}\right)=\varphi_{k, k}\left(c_{i, k}-c_{j, k}\right)$. Hence, for $k=1$ and $j \neq 1$ we obtain

$$
c_{i, j}=\varphi_{j, 1}\left(c_{i, j}\right)=\varphi_{1,1}\left(c_{i, 1}-c_{j, 1}\right)
$$

In particular, for $j \neq 1$ and $i=1$

$$
c_{1, j}=\varphi_{1,1}\left(c_{1,1}-c_{j, 1}\right)=-\varphi_{1,1}\left(c_{j, 1}\right) .
$$

Let $\varphi:=\varphi_{1,1}$. Corollary 6.6 and Lemma 6.8 directly imply
Lemma 6.9. For each $1 \neq i \in I, A_{i}=\varphi\left(A_{1}\right)$.
Proof. Let $1 \neq j \in I$. By indecomposability, for each $1 \neq i \in I$, the group $A_{i}$ is generated by sets: $\varphi_{1, i}\left(A_{1}\right)=\varphi^{2}\left(A_{1}\right), \varphi_{j, i}\left(A_{j}\right)=\varphi\left(A_{j}\right)$ and all elements $c_{1, i}=-\varphi\left(c_{i, 1}\right)$ and $c_{j, i}=\varphi\left(c_{j, 1}-c_{i, 1}\right)$. Since each $A_{j}$ is a subgroup of $A_{1}$, it it evident that $A_{i} \subseteq \varphi\left(A_{1}\right)$.

On the other hand, the group $A_{1}$ is generated by sets: $\varphi_{1,1}\left(A_{1}\right)=\varphi\left(A_{1}\right), \varphi_{j, 1}\left(A_{j}\right)=A_{j}$ and all constants $c_{j, 1}$. Hence, $\varphi\left(A_{1}\right)$ is generated by $\varphi^{2}\left(A_{1}\right), \varphi\left(A_{j}\right)$ and $\varphi\left(c_{j, 1}\right)$, which shows that $A_{i}=\varphi\left(A_{1}\right)$ for each $1 \neq i \in I$.
Lemma 6.10. For $i \neq j, c_{i, 1}-c_{j, 1} \notin \varphi\left(A_{1}\right)$.
Proof. Assume $c_{i, 1}-c_{j, 1} \in \varphi\left(A_{1}\right)=A_{j}$, for some $i \neq j \in I$. Then there exists $a \in A_{j}$ such that $c_{i, 1}=c_{j, 1}+a$. But, for each $b \in A_{1}, 0_{i} \cdot b=(1-\varphi)(b)+c_{i, 1}=(1-\varphi)(b)+a+c_{j, 1}=a \cdot b$. Then, by Lemma 6.5, $0_{i}=a$, a contradiction with Lemma 6.7.

Lemma 6.10 gives the upper bound for the number of orbits in a non-connected SI quasi-reductive medial quandle.
Corollary 6.11. Let $\kappa=\left|A_{1} / \varphi\left(A_{1}\right)\right|$. The number of orbits in $Q$ is at most $\kappa+1$.
Now we are ready to describe the structure of non-connected SI quasi-reductive medial quandles.
Theorem 6.12. Let $Q$ be a non-connected quasi-reductive medial quandle. Then $Q$ is subdirectly irreducible if and only if it is isomorphic to the sum of the affine mesh

$$
((A, \underbrace{\varphi(A), \varphi(A), \ldots)}_{\nu-\text { times }} ;\left(\begin{array}{cccccccccc}
\varphi & \varphi^{2} & \varphi^{2} & \ldots & \varphi^{2} & \ldots \\
1 & \varphi & \varphi & \ldots & \varphi \\
1 & \varphi & \varphi & \ldots & \varphi \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \varphi & \varphi & \ldots & \varphi & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right) ;\left(\begin{array}{ccccccc}
0 & -\varphi\left(c_{2,1}\right) & \ldots & -\varphi\left(c_{i, 1}\right) & \ldots & -\varphi\left(c_{j, 1}\right) & \ldots \\
c_{2,1} & 0 & \ldots & \varphi\left(c_{2,1}-c_{i, 1}\right) & \ldots & \varphi\left(c_{2,1}-c_{j, 1}\right. & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_{i, 1} & \varphi\left(c_{i, 1}-c_{2,1}\right) & \ldots & 0 & \ldots & \varphi\left(c_{j, 1}-c_{i, 1}\right) & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_{j, 1} & \varphi\left(c_{j, 1}-c_{2,1}\right) & \ldots & \varphi\left(c_{j, 1}-c_{i, 1}\right) & \ldots & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right)),
$$

where
(1) $A$ is a subdirectly irreducible $\mathbb{Z}\left[x, x^{-1}\right]$-module,
(2) $\varphi=1-x$,
(3) $0<\nu \leq \kappa$, where $\kappa=|A / \varphi(A)|$,
(4) $c_{i, 1}-c_{j, 1} \notin \varphi(A)$, for each $1<i \neq j \in I$,
(5) $A$ is generated by the set $\varphi(A) \cup\left\{c_{i, 1} \mid i \in I\right\}$.

Proof. It is easy to check that $\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$ is an irreducible affine mesh.
" $\Rightarrow$ " Let $Q$ be a non-connected SI quasi-reductive medial quandle. The proof that $Q$ has the form described in the theorem follows from Lemmas 6.4, 6.7, 6.8, 6.9, 6.10, Corollaries 6.6, and 6.11.
" $\Leftarrow$ " Now let $Q$ be a sum of the affine mesh described in the theorem. Then, by assumption, $A_{1}=A$ is a subdirectly irreducible $\mathbb{Z}\left[x, x^{-1}\right]$-module. Let $M$ be the smallest non-trivial submodule of the module $A_{1}$. Since $\operatorname{Ker}(\varphi)$ is a non-zero submodule of $A$, clearly $M \subseteq \operatorname{Ker}(\varphi)$. By Example 3.4, the relation $\Upsilon \subseteq Q \times Q$ defined as follows:

$$
a \Upsilon b \text { if and only if } a=b \text { or }\left(a, b \in A_{1} \text { and } a \equiv_{M} b\right)
$$

is a congruence of the quandle $Q$.
To prove that $Q$ is subdirectly irreducible we will show that for any $a \neq b \in Q$ the congruence $\Theta(a, b)$ generated by $a$ and $b$ contains the congruence $\Upsilon$.

It is obvious for $a, b \in A_{1}$. Now we will show that for $a$ or $b$ in $Q \backslash A_{1}$, the congruence $\left.\Theta(a, b)\right|_{A_{1}}$ is non-trivial. We will divide the proof into several cases.

Case 1. Let $a, b \in A_{i}=\varphi(A)$ for $1 \neq i \in I$. It is easy to notice that for any $x \in A_{1}$

$$
\begin{array}{r}
a \cdot x=c_{i, 1}+\varphi_{i, 1}(a)+\left(1-\varphi_{1,1}\right)(x)=c_{i, 1}+a+\left(1-\varphi_{1,1}\right)(x) \neq \\
c_{i, 1}+b+\left(1-\varphi_{1,1}\right)(x)=c_{i, 1}+\varphi_{i, 1}(b)+\left(1-\varphi_{1,1}\right)(x)=b \cdot x .
\end{array}
$$

Case 2. Let $a \in A_{i}=\varphi(A)$ and $b \in A_{j}=\varphi(A)$ for $1<i \neq j \in I$. Then there are $a_{1}, b_{1} \in A_{1}$ such that $a=\varphi\left(a_{1}\right)$ and $b=\varphi\left(b_{1}\right)$. Furthermore, by the assumption, the constants $c_{i, 1}$ and $c_{j, 1}$ belong to different cosets of $\varphi(A)$, and hence we have that $c_{i, 1} \notin c_{j, 1}+\varphi(A)$. This implies that $c_{i, 1} \neq c_{j, 1}+\varphi\left(b_{1}\right)-\varphi\left(a_{1}\right)$. Hence for any $x \in A_{1}$

$$
\begin{aligned}
a \cdot x= & c_{i, 1}+\varphi_{i, 1}(a)+\left(1-\varphi_{1,1}\right)(x)=c_{i, 1}+a+\left(1-\varphi_{1,1}\right)(x)=c_{i, 1}+\varphi\left(a_{1}\right)+\left(1-\varphi_{1,1}\right)(x) \neq \\
c_{j, 1} & +\varphi\left(b_{1}\right)+\left(1-\varphi_{1,1}\right)(x)=c_{j, 1}+b+\left(1-\varphi_{1,1}\right)(x)=c_{j, 1}+\varphi_{j, 1}(b)+\left(1-\varphi_{1,1}\right)(x)=b \cdot x .
\end{aligned}
$$

Case 3. Let $a \in A_{1}, b \in A_{i}=\varphi(A)$ for $1 \neq i \in I$ and $c_{i, 1} \notin \varphi(A)$. Then there is $b_{1} \in A_{1}$ such that $b=\varphi\left(b_{1}\right)$ and $c_{i, 1} \neq \varphi(a)-\varphi\left(b_{1}\right)$. In consequence, for any $x \in A_{1}$, we obtain

$$
\begin{array}{r}
a \cdot x=\varphi_{1,1}(a)+\left(1-\varphi_{1,1}\right)(x)=\varphi(a)+\left(1-\varphi_{1,1}\right)(x) \neq \\
c_{i, 1}+\varphi\left(b_{1}\right)+\left(1-\varphi_{1,1}\right)(x)=c_{i, 1}+b+\left(1-\varphi_{1,1}\right)(x)= \\
c_{i, 1}+\varphi_{i, 1}(b)+\left(1-\varphi_{1,1}\right)(x)=b \cdot x .
\end{array}
$$

Case 4. Let $a \in A_{1}, b \in A_{i}=\varphi(A)$ for $1 \neq i \in I$ and $c_{i, 1} \in \varphi(A)$. Since, by assumption, the group $A$ is generated by the set $\varphi(A) \cup\left\{c_{i, 1} \mid i \in I\right\}$, there is $j \in I$, such that $1 \neq j \neq i$, with $c_{j, 1} \notin \varphi(A)$. Then $c_{j, 1} \neq \varphi^{2}(a)-\varphi\left(c_{i, 1}-c_{j, 1}\right)-\varphi(b)$. Hence

$$
\begin{array}{r}
a \cdot 0_{j}=c_{1, j}+\varphi_{1, j}(a)+\left(1-\varphi_{j, j}\right)\left(0_{j}\right)=c_{1, j}+\varphi_{1, j}(a)=-c_{j, 1}+\varphi^{2}(a) \neq \\
\varphi\left(c_{i, 1}-c_{j, 1}\right)+\varphi(b)=c_{i, j}+\varphi_{i, j}(b)=c_{i, j}+\varphi_{i, j}(b)+\left(1-\varphi_{j, j}\right)\left(0_{j}\right)=b \cdot 0_{j} .
\end{array}
$$

Since $a \cdot 0_{j}, b \cdot 0_{j} \in A_{j}$, by Case 1 we have that for any $x \in A_{1},\left(a \cdot 0_{j}\right) \cdot x \neq\left(b \cdot 0_{j}\right) \cdot x$.
Hence, by Cases $1-4$, for any $a \neq b$ with $a$ or $b$ in $Q \backslash A_{1},\left.\Theta(a, b)\right|_{A_{1}}$ is indeed non-trivial, which shows that $Q$ is subdirectly irreducible.

Corollary 6.13. Let $Q$ be a subdirectly irreducible m-reductive medial quandle. Then

- $|Q|=2$, if $m=1$;
- $Q$ is constructed in Theorem 6.12 where $A$ is a subdirectly irreducible $\mathbb{Z}\left[x, x^{-1}\right] /(1-x)^{m-1}$ module, if $m \geq 2$.

Example 6.14. Let $A=\mathbb{Z}_{p^{\infty}} \times \mathbb{Z}_{p^{\infty}}$ and let $x \cdot(a, b)=(a, b-a)$. Then $A$ is a subdirectly irreducible $\mathbb{Z}\left[x, x^{-1}\right]$-module since the smallest submodule is $\left\{(0, c / p)\right.$, for $\left.c \in \mathbb{Z}_{p}\right\}$. Let now $\varphi$ be
the multiplication by $1-x$, i.e. $\varphi((a, b))=(0, a)$. Clearly $\varphi^{2}=0$. We obtain a SI medial reductive quandle with infinitely many orbits as the sum of

$$
\left(\left(\mathbb{Z}_{p^{\infty}} \times \mathbb{Z}_{p^{\infty}}, \mathbb{Z}_{p \infty}, \mathbb{Z}_{p \infty}, \ldots\right) ;\left(\begin{array}{cccc}
\varphi & 0 & 0 & \cdots \\
1 & \varphi & \varphi & \cdots \\
1 & \varphi & \varphi & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}\right) ;\left(\begin{array}{cccc}
0 & (0,-1 / p) & \left(0,-1 / p^{2}\right) & \cdots \\
(1 / p, 0) & 0 & \left(0,(p-1) / p^{2}\right) & \cdots \\
\left(1 / p^{2}, 0\right) & \left(0,(1-p) / p^{2}\right) & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}\right)\right)
$$

It remains to decide which of the sums of affine meshes described in Theorem 6.12 are isomorphic. Let us start with homologous affine meshes introduced in [7].
Definition 6.15. We call two affine meshes $\mathcal{A}=\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$ and $\mathcal{A}^{\prime}=\left(A_{i}^{\prime} ; \varphi_{i, j}^{\prime} ; c_{i, j}^{\prime}\right)$, over the same index set $I$, homologous, if there is a permutation $\sigma$ of the set $I$, group isomorphisms $\psi_{i}$ : $A_{i} \rightarrow A_{\sigma i}^{\prime}$, and constants $d_{i} \in A_{\sigma i}^{\prime}$, such that, for every $i, j \in I$,
(H1) $\psi_{j} \varphi_{i, j}=\varphi_{\sigma i, \sigma j}^{\prime} \psi_{i}$, i.e., the following diagram commutes:

(H2) $\psi_{j}\left(c_{i, j}\right)=c_{\sigma i, \sigma j}^{\prime}+\varphi_{\sigma i, \sigma j}^{\prime}\left(d_{i}\right)-\varphi_{\sigma j, \sigma j}^{\prime}\left(d_{j}\right)$.
Theorem 6.16. [7] Let $\mathcal{A}=\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$ and $\mathcal{A}^{\prime}=\left(A_{i}^{\prime} ; \varphi_{i, j}^{\prime} ; c_{i, j}^{\prime}\right)$ be two indecomposable affine meshes, over the same index set $I$. Then the sums of $\mathcal{A}$ and $\mathcal{A}^{\prime}$ are isomorphic quandles if and only if the meshes $\mathcal{A}, \mathcal{A}^{\prime}$ are homologous.

Let $\mathcal{A}=\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$ be an indecomposable affine mesh described in Theorem 6.12 . We have the following sequence of lemmas.

Lemma 6.17. For each $i \in I$, let $c_{i, 1}^{\prime} \in A$ be such that $c_{i, 1}^{\prime} \in c_{i, 1}+\varphi(A)$. Then the sum of $\mathcal{A}$ is isomorphic to the sum of the indecomposable affine mesh $\mathcal{A}^{\prime}=\left(A_{i} ; \varphi_{i, j} ; c_{i, j}^{\prime}\right)$.
Proof. Let $\sigma=i d, \psi_{1}=i d, d_{1}=0$ and for every $1 \neq i \in I, \psi_{i}=i d$ and $d_{i}=c_{i, 1}-c_{i, 1}^{\prime} \in \varphi(A)=A_{i}$. Hence, condition (H1) is satisfied trivially. Moreover,

$$
c_{i, 1}=c_{i, 1}^{\prime}+d_{i}-0=c_{i, 1}^{\prime}+\varphi_{i, 1}\left(d_{i}\right)-\varphi_{1,1}(0)
$$

which shows that the condition (H2) is also satisfied.
Lemma 6.18. Let $A / \varphi(A)$ be a cyclic group and $\kappa=|A / \varphi(A)|>1$. Then

- there is exactly one, up to isomorphism, SI quasi-reductive medial quandle with two orbits;
- there is exactly one SI quasi-reductive medial quandle with three orbits, such that $c_{3,1} \in$ $\varphi(A)$;
- if $\kappa<\omega$ then there is exactly one SI quasi-reductive medial quandle with $\kappa+1$ orbits.

Proof. Since $A$ is generated by the set $\varphi(A) \cup\left\{c_{i, 1} \mid i \in I\right\}$, at least one constant must be a generator of the group $A / \varphi(A)$. Hence, if $Q$ has only two orbits, the constant $c=c_{2,1} \notin \varphi(A)$ and $c+\varphi(A)$ is one of the generators of $A / \varphi(A)$. Therefore, there exists an isomorphism $\psi$ : $A \rightarrow A$ such that, for any other $d \notin \varphi(A)$, where $d+\varphi(A)$ is a generator of $A / \varphi(A)$, we have $\psi(c)=d$. Hence, for $\sigma=i d, \psi_{1}=\psi_{2}=\psi$ and $d_{1}=d_{2}=0$, the conditions (H1) and (H2) are satisfied for affine meshes: $\left((A, \varphi(A)) ;\left(\begin{array}{cc}\varphi & \varphi^{2} \\ 1 & \varphi\end{array}\right) ;\left(\begin{array}{cc}0 & -\varphi(c) \\ c & 0\end{array}\right)\right)$ and $\left((A, \varphi(A)) ;\left(\begin{array}{cc}\varphi & \varphi^{2} \\ 1 & \varphi\end{array}\right) ;\left(\begin{array}{cc}0 & -\varphi(d) \\ d & 0\end{array}\right)\right)$. In consequence, they are isomorphic.

The same arguments works in the case of three orbits with $c_{3,1}=0$. So by Lemma 6.17 all SI quasi-reductive medial quandles with 3 orbits, where $c_{3,1} \in \varphi(A)$ are isomorphic.

Finally, in the case of $\kappa+1$ orbits, the required condition that, for each $1<i \neq j \in I$, $c_{i, 1} \notin c_{j, 1}+\varphi(A)$, implies (by Lemma 6.17) that there is only one way for choosing constants in $A$. So, the statement is obvious.

Lemma 6.19. Suppose that for some $1 \neq i \in I, c_{i, 1}=0$. Then the sum of $\mathcal{A}$ is not isomorphic to the sum of the indecomposable affine mesh $\mathcal{A}^{\prime}=\left(A_{i} ; \varphi_{i, j} ; c_{i, j}^{\prime}\right)$ with $c_{i, 1}^{\prime} \notin \varphi(A)$ for each $1 \neq i \in I$.
Proof. Let $c_{i, 1}=0$ for some $1 \neq i \in I$. Since $0 \in \varphi(A)$ and $c_{i, j}^{\prime}+\varphi(A) \neq \varphi(A)$, for any isomorphism $\psi: A \rightarrow A$, every $d_{1} \in A$ and $d_{i} \in \varphi(A)$, we have

$$
\psi\left(c_{i, 1}\right)=0 \neq c_{i, j}^{\prime}+\varphi_{i, 1}\left(d_{i}\right)-\varphi_{1,1}\left(d_{1}\right)=c_{i, j}^{\prime}+d_{i}-\varphi\left(d_{1}\right) \in c_{i, j}^{\prime}+\varphi(A)
$$

This means that the condition (H2) fails for any isomorphism $\psi: A \rightarrow A$.
By Lemmas 6.17 and 6.19 we immediately obtain
Corollary 6.20. Let $c_{i, 1} \in \varphi(A)$ for some $1 \neq i \in I$. Then the sum of $\mathcal{A}$ is not isomorphic to the sum of the indecomposable affine mesh $\mathcal{A}^{\prime}=\left(A_{i} ; \varphi_{i, j} ; c_{i, j}^{\prime}\right)$ with $c_{i, 1}^{\prime} \notin \varphi(A)$ for each $1 \neq i \in I$.
Example 6.21. Let $\varphi=\left(\begin{array}{ll}0 & 0 \\ 1 & 0\end{array}\right): \mathbb{Z}_{2} \times \mathbb{Z}_{2} \rightarrow \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ and $c=\binom{1}{0}$.
It was shown in [7], that up to isomorphism, there are exactly two reductive, but not 2-reductive medial quandles of size 6: $\left(\left(\mathbb{Z}_{2^{2}}, 2 \mathbb{Z}_{2^{2}}\right) ;\left(\begin{array}{cc}2 & 0 \\ 1 & 2\end{array}\right) ;\left(\begin{array}{cc}0 & -2 \\ 1 & 0\end{array}\right)\right)$ and $\left(\left(\mathbb{Z}_{2} \times \mathbb{Z}_{2}, \varphi\left(\mathbb{Z}_{2} \times \mathbb{Z}_{2}\right)\right) ;\left(\begin{array}{cc}\varphi & 0 \\ 1 & \varphi\end{array}\right) ;\left(\begin{array}{cc}0 & -\varphi(c) \\ c & 0\end{array}\right)\right)$. By Theorem 6.12 both of them are subdirectly irreducible.

Further, there are nine reductive, but not 2-reductive medial quandles of size 8 . Only two of them are subdirectly irreducible: $\left(\left(\mathbb{Z}_{2^{2}}, 2 \mathbb{Z}_{2^{2}}, 2 \mathbb{Z}_{2^{2}}\right) ;\left(\begin{array}{ccc}2 & 0 & 0 \\ 1 & 2 & 2 \\ 1 & 2\end{array}\right) ;\left(\begin{array}{ccc}0 & -2 & 0 \\ 1 & 0 & 2 \\ 0 & -2 & 0\end{array}\right)\right)$ and
$\left(\left(\mathbb{Z}_{2} \times \mathbb{Z}_{2}, \varphi\left(\mathbb{Z}_{2} \times \mathbb{Z}_{2}\right), \varphi\left(\mathbb{Z}_{2} \times \mathbb{Z}_{2}\right)\right) ;\left(\begin{array}{ccc}\varphi & 0 & 0 \\ 1 & \varphi & \varphi \\ 1 & \varphi & \varphi\end{array}\right) ;\left(\begin{array}{ccc}0 & -\varphi(c) & 0 \\ c & 0 & \varphi(c) \\ 0 & -\varphi(c) & 0\end{array}\right)\right)$.
Note that all of them are strictly 3 -reductive.

## 7. Subdirectly irreducible quandles with cyclic orbits

It is clear that each congruence of a cyclic group $A$ is a congruence of the $\mathbb{Z}\left[x, x^{-1}\right]$-module $A$. Consequently, a $\mathbb{Z}\left[x, x^{-1}\right]$-module $A$ with the underlying group cyclic is subdirectly irreducible if and only if $A$ is subdirectly irreducible as an abelian group. The only cyclic SI groups are groups $\mathbb{Z}_{p^{s}}$ of order $p^{s}$, for some prime number $p$.

Moreover, the only non-zero nilpotent endomorphisms of the group $\mathbb{Z}_{p^{s}}$ are of the form $\varphi=p^{k} a$, for some $0<k<s$ and $a$ coprime with $p$, and the quandles $\operatorname{Aff}\left(\mathbb{Z}_{p^{s}}, 1-p^{k} a\right)$ are strictly ( $\left.\left\lceil\frac{s}{k}\right\rceil+1\right)$ reductive.

Consider now 2-reductive medial quandles. Since $\mathbb{Z}\left[x, x^{-1}\right] /(1-x) \cong \mathbb{Z}$ and each finite subdirectly irreducible $\mathbb{Z}$-module is a cyclic group, Theorem 6.12 immediately gives the known characterization of finite SI 2-reductive medial quandles which was presented by Romanowska and Roszkowska in [11].

Theorem 7.1. [11, Theorem 3.1] A finite strictly 2-reductive medial quandle $Q$ is subdirectly irreducible if and only if $Q$ is isomorphic to the sum of an affine mesh

$$
((\mathbb{Z}_{p^{k}}, \underbrace{\mathbb{Z}_{1}, \ldots, \mathbb{Z}_{1}}_{n \text {-times }}) ; 0 ;\left(c_{i, j}\right)),
$$

where $p^{k}$, for $k>0$, is a prime power, $1 \leq n \leq p^{k}$, and $c_{i, 1} \in \mathbb{Z}_{p^{k}}$ are pairwise different elements such that $\mathbb{Z}_{p^{k}}=\left\langle c_{i, 1}: i \in I\right\rangle$.

Moreover, we are now able to describe all SI 2-reductive quandles.

Theorem 7.2. All infinite subdirectly irreducible 2-reductive medial quandles are sums of

$$
((\mathbb{Z}_{p^{\infty}}, \underbrace{\mathbb{Z}_{1}, \mathbb{Z}_{1}, \ldots}_{\omega-\text { times }}) ; 0 ;\left(c_{i, j}\right))
$$

where $p$ is a prime and $c_{i, 1}$ are pairwise different elements of $\mathbb{Z}_{p} \infty$. There is $2^{\omega}$ isomorphism classes of such quandles.
Proof. Consider $Q$ an infinite subdirectly irreducible 2-reductive medial quandle. We construct $Q$ as the sum of the mesh from Theorem 6.12. If $A_{1}$ is finite than the number of orbits is finite too. Hence $A_{1}$ is an infinite SI abelian group. According to [2, Theorem 3.29] such are only Prüfer groups $\mathbb{Z}_{p \infty}$. Since these groups are not finitely generated, there has to be infinitely many constants. Any infite subset of $\mathbb{Z}_{p \infty}$ already generates the group.

There is $2^{\omega}$ possibilities how to choose the set $\left\{c_{i, 1} \mid i \in I\right\}$. According to Theorem 6.16, each such a mesh is homologous to at most $\left|\operatorname{Aut}\left(\mathbb{Z}_{p^{\infty}}\right)\right|$ meshes since the constants $d_{i}$ play no role here. And $\left|\operatorname{Aut}\left(\mathbb{Z}_{p} \infty\right)\right|=\omega$.

A binary algebra $Q$ is called involutory if $L_{a}^{2}=1$, for every $a \in Q$, i.e., if it satisfies

$$
x(x y)=y .
$$

We know (see [7, Proposition 7.2]) that an involutory medial quandle has orbits that are modules over $\mathbb{Z}[x] /(1+x) \cong \mathbb{Z}$ and therefore, when considering congruences, we just look at subgroups. Hence, in a SI finite involutory medial quandle, each orbit is cyclic. Moreover, since $1-x \equiv_{(1+x)}$ 2 then Theorem 6.12 confirms the result of Roszkowska:

Theorem 7.3. [13, Theorem 4.3] A finite involutory and reductive medial quandle $Q$ is subdirectly irreducible if and only if $Q$ is isomorphic to the sum of one of the following affine meshes:

$$
\left(\left(\mathbb{Z}_{2^{k}}, 2 \mathbb{Z}_{2^{k}}\right) ;\left(\begin{array}{cc}
2 & 4 \\
1 & 2
\end{array}\right) ;\left(\begin{array}{cc}
0 & -2 \\
1 & 0
\end{array}\right)\right), \quad\left(\left(\mathbb{Z}_{2^{k}}, 2 \mathbb{Z}_{2^{k}}, 2 \mathbb{Z}_{2^{k}}\right) ;\left(\begin{array}{ccc}
2 & 4 & 4 \\
1 & 2 & 2 \\
1 & 2 & 2
\end{array}\right) ;\left(\begin{array}{ccc}
0 & -2 & 0 \\
1 & 0 & 2 \\
0 & -2 & 0
\end{array}\right)\right),
$$

where $k \geq 1$.
Moreover, we can tell something about infinite subdirectly irreducibles.
Theorem 7.4. An infinite involutory, non-connected and quasi-reductive medial quandle $Q$ is subdirectly irreducible if and only if $Q$ is isomorphic to the sum of

$$
\left(\left(\mathbb{Z}_{2^{\infty}}, \mathbb{Z}_{2^{\infty}}\right) ;\left(\begin{array}{c}
2 \\
1 \\
1
\end{array} 2\right) ;\left(\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right)\right) .
$$

Proof. The only infinite subdirectly irreducible abelian group, where the multiplication by 2 is not $1-1$, is $\mathbb{Z}_{2^{\infty}}$. Since $2 \cdot \mathbb{Z}_{2^{\infty}}=\mathbb{Z}_{2^{\infty}}$, there are at most two orbits and the constants are not important, according to Lemma 6.17.

Clearly, each pair of endomorphisms of a cyclic group conjugates if and only if they are equal. Hence, by Theorem 6.16, for a group $\mathbb{Z}_{p^{s}}$ and two different nilpotent endomorphisms of $\mathbb{Z}_{p^{s}}$ we always obtain non-isomorphic quandles. So, in non-isomorphic sums of affine meshes with cyclic orbits, constants must play a crucial role.

Now we give the characterization of non-isomorphic SI finite reductive medial quandles with each orbit cyclic.

Theorem 7.5. Let $n \geq 1, I=\{1,2, \ldots, n, n+1\}$ and $K=\{2, \ldots, n, n+1\}$. Let $\mathcal{A}=\left(A_{i} ; \varphi_{i, j} ; c_{i, j}\right)$ and $\mathcal{A}^{\prime}=\left(A_{i} ; \varphi_{i, j} ; c_{i, j}^{\prime}\right)$ be two indecomposable affine meshes over I described in Theorem 6.12 with $A=\mathbb{Z}_{p^{s}}$, for some prime power $p^{s}$, and $\varphi=p^{k} a$, for some $0<k<s$ and a coprime with $p$. Assume that $c_{i, 1}, c_{i, 1}^{\prime} \notin \varphi(A)$, for each $i \in K$, or there is (exactly one) $i \in K$ such that $c_{i, 1}, c_{i, 1}^{\prime} \in \varphi(A)$.

Then the sums $\mathcal{A}$ and $\mathcal{A}^{\prime}$ are isomorphic if and only if $n \leq\left\lceil\frac{s}{k}\right\rceil$ or there is a permutation $\sigma$ of the set $K$ such that, for any $i, j \in K$, the constants satisfy the following condition:

$$
\begin{equation*}
c_{i, 1} c_{\sigma(j), 1}^{\prime}-c_{j, 1} c_{\sigma(i), 1}^{\prime}=0 \tag{7.1}
\end{equation*}
$$

Proof. By Theorem 6.16, two indecomposable affine meshes over the same index set $I$ are isomorphic if and only if the meshes are homologous. Hence, to show that the meshes $\mathcal{A}$ and $\mathcal{A}^{\prime}$ are isomorphic it is enough to check the condition (H2) only for constants $c_{i, 1} \in A, i \in I$, (the condition (H1) is trivially satisfied). So, we have to check whether there are a permutation $\sigma$ of the set $K$, a group isomorphism $\psi: A \rightarrow A$ and constants $d_{1} \in A$ and $d_{i} \in A_{\sigma(i)}$ such that for every $i \in K$,

$$
\psi\left(c_{i, 1}\right)=c_{\sigma(i), 1}^{\prime}+\varphi_{\sigma(i), 1}\left(d_{i}\right)-\varphi_{1,1}\left(d_{1}\right)=c_{\sigma(i), 1}^{\prime}+d_{i}-\varphi\left(d_{1}\right) .
$$

Since for $i \neq 1, d_{i} \in \varphi(A)$, then there is $a_{i} \in A$ such that $d_{i}=\varphi\left(a_{i}\right)$ and $d_{i}-\varphi\left(d_{1}\right)=\varphi\left(a_{i}-d_{1}\right) \in$ $\varphi(A)$. Therefore our problem can be reformulated in the following way: Are there a permutation $\sigma$ of the set $K$, a group isomorphism $\psi: A \rightarrow A$ and constants $r_{i} \in \varphi(A)$ such that for every $i \in K$,

$$
\psi\left(c_{i, 1}\right)=c_{\sigma(i), 1}^{\prime}+r_{i} ?
$$

The condition $r \in \varphi\left(\mathbb{Z}_{p^{s}}\right)$ is equivalent to the fact that there is $z \in \mathbb{Z}_{p^{s}}$ such that $r=p^{k} z$. Further, each isomorphism of the group $\mathbb{Z}_{p^{s}}$ is defined in the way: $1 \mapsto y+p^{l} b$, where $y \in\left\{1, \ldots, p^{l}-1\right\}$ and $b \in\left\{0, \ldots, p^{l}-1\right\}$.

Hence, the problem reduces to the question about existing solutions of the following system of $n$ linear equations:

$$
\begin{equation*}
c_{i, 1} y+c_{i, 1} p^{l} b+p^{k} x_{i}=c_{\sigma(i), 1}^{\prime}, \tag{7.2}
\end{equation*}
$$

with $2 \leq i \leq n+1$ and ( $n+2$ ) unknowns: $y \in\left\{1, \ldots, p^{l}-1\right\}, b \in\left\{0, \ldots, p^{l}-1\right\}$ and $x_{2}, \ldots, x_{n+1} \in$ $\mathbb{Z}_{p^{s}}$, for some permutation $\sigma$ of the set $K$.

$$
\text { Let } B=\left(\begin{array}{cccccc}
c_{2,1} & c_{2,1} p^{l} & p^{k} & 0 & \ldots & 0 \\
c_{3,1} & c_{3,1} p^{l} & 0 & p^{k} & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_{n+1,1} & c_{n+1,1 p} & 0 & 0 & \ldots & p^{k}
\end{array}\right) \text { and } C=\left(\begin{array}{c}
c_{\sigma(2), 1}^{\prime} \\
c_{\sigma(3), 1}^{\prime} \\
\vdots \\
c_{\sigma(n+1), 1}^{\prime}
\end{array}\right) \text {. }
$$

The system (7.2) is solvable if and only if $r k(B)=r k(B \mid C)$, where $r k$ denotes the rank of a matrix. Let $m=\left\lceil\frac{s}{k}\right\rceil$. Since the sums $\mathcal{A}$ and $\mathcal{A}^{\prime}$ are strictly $(m+1)$-reductive and there is $i \in K$ such that $c_{i, 1} \notin \varphi\left(\mathbb{Z}_{p^{s}}\right)$, then in the case $n \leq m, r k(B)=r k(B \mid C)=n$ and the system (7.2) always has a solution.

On the other hand, if $n>m$, then $\operatorname{rk}(B)=m$. In this case, the system has a solution if and only if there is a permutation $\sigma$ of the set $K$ and $c_{i, 1} c_{\sigma(j), 1}^{\prime}-c_{j, 1} c_{\sigma(i), 1}^{\prime}=0$ for any $i, j \in K$. This completes the proof.

Example 7.6. Using Theorem 7.5 it is easy to check that the sums of the following two strictly 3 -reductive affine meshes with 4 orbits:
$\left(\left(\mathbb{Z}_{7^{2}}, 7 \mathbb{Z}_{7^{2}}, 7 \mathbb{Z}_{7^{2}}, 7 \mathbb{Z}_{7^{2}}\right) ;\left(\begin{array}{ccc}7 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 7 & 7 \\ 1 & 7 & 7 \\ 1 & 7 & 7\end{array}\right) ;\left(\begin{array}{cccc}0 & 42 & 28 & 21 \\ 1 & 0 & 25 & 28 \\ 3 & 14 & 0 & 28 \\ 4 & 21 & 7 & 4\end{array}\right)\right), \quad\left(\left(\mathbb{Z}_{7^{2}}, 7 \mathbb{Z}_{7^{2}}, 7 \mathbb{Z}_{7^{2}}, 7 \mathbb{Z}_{7^{2}}\right) ;\left(\begin{array}{cccc}7 & 0 & 0 & 0 \\ 1 & 7 & 7 & 7 \\ 1 & 7 & 7 \\ 1 & 7 & 7 & 7\end{array}\right) ;\left(\begin{array}{cccc}0 & 14 & 35 & 7 \\ 5 & 0 & 21 & 7 \\ 2 & 28 & 0 & 42 \\ 6 & 7 & 28 & 0\end{array}\right)\right)$
are not isomorphic. But their lattices of congruences are (to compute the lattice of congruences we used [3]).

On the other hand, for each module $\mathbb{Z}_{p^{s}}$ such that $\varphi^{2}\left(\mathbb{Z}_{p^{s}}\right)=0$ and $\left|\mathbb{Z}_{p^{s}} / \varphi\left(\mathbb{Z}_{p^{s}}\right)\right| \geq 3$, there are exactly two non-isomorphic sums of the following affine meshes with 3 orbits:
$\left(\left(\mathbb{Z}_{p^{s}}, \varphi\left(\mathbb{Z}_{p^{s}}\right), \varphi\left(\mathbb{Z}_{p^{s}}\right)\right) ;\left(\begin{array}{ccc}\varphi & 0 & 0 \\ 1 & \varphi & \varphi \\ 1 & \varphi & \varphi\end{array}\right) ;\left(\begin{array}{ccc}0 & -\varphi(1) & 0 \\ 1 & 0 & \varphi(1) \\ 0 & -\varphi(1) & 0\end{array}\right)\right)$ and
$\left(\left(\mathbb{Z}_{p^{s}}, \varphi\left(\mathbb{Z}_{p^{s}}\right), \varphi\left(\mathbb{Z}_{p^{s}}\right)\right) ;\left(\begin{array}{ccc}\varphi & 0 & 0 \\ 1 & \varphi & \varphi \\ 1 & \varphi & \varphi\end{array}\right) ;\left(\begin{array}{ccc}0 & -\varphi(1) & -\varphi(c) \\ 1 & 0 & \varphi(1-c) \\ c & \varphi(c-1) & 0\end{array}\right)\right)$, where $c \in \mathbb{Z}_{p^{s}} \backslash \varphi\left(\mathbb{Z}_{p^{s}}\right)$.

## 8. Discussion

The optimal outcome of our work would be a complete classification of subdirectly irrreducible medial quandles. We have not achieved the goal so far, what we know is that

- Subdirectly irreducible medial quandles are either quasi-affine or of set type.
- Finite quasi-affine quandles are connected affine quandles. Classification of connected affine quandles depends on the classification of subdirectly irreducible $\mathbb{Z}\left[x, x^{-1}\right]$-modules. This classification is still open.
- It is unknown whether there exist infinite quasi-affine SI medial quandles that are not connected. Proposition 4.5 only excludes the possibility of affine non-connected ones.
- In the finite case, all SI medial quandles of set type are reductive and they are described in Corollary 6.13 modulo the missing description of irreducible $\mathbb{Z}\left[x, x^{-1}\right]$-modules.
- Infinite SI medial quandles of set type need not be reductive, as we saw in Theorem 7.4. Here we used an auxiliary notition called quasi-reductivity but there is no evidence whether this definition includes all SI medial quandles of set type, that means whether or not there exist infinite SI medial quandles that are neither quasi-affine nor quasi-reductive.
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(P.J.) Department of Mathematics, Faculty of Engineering, Czech University of Life Sciences, Kamýcká 129, 16521 Praha 6, Czech Republic
(A.P., A.Z.) Faculty of Mathematics and Information Science, Warsaw University of Technology, Koszykowa 75, 00-662 Warsaw, Poland

E-mail address: (P.J.) jedlickap@tf.czu.cz
E-mail address: (A.P.) apili@mini.pw.edu.pl
E-mail address: (A.Z.) A.Zamojska-Dzienio@mini.pw.edu.pl


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