THE LATTICE OF QUASIVARIETES OF MODULES OVER A DEDEKIND RING

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Abstract. In 1995 D. V. Belkin described the lattice of quasivarieties of modules over principal ideal domains [1]. The following paper provides a description of the lattice of subquasivarieties of the variety of modules over a given Dedekind ring. It also shows which subvarieties of these modules are deductive (a variety is deductive if every subquasivariety is a variety).

1. Introduction

Until recently, the only known facts concerning quasivarieties of modules were provided by Belkin [1] who characterised all quasivarieties of modules over principal ideal domains, effectively generalizing the result of Vinogradov [12].

A natural question arises here: does Belkin's result apply to principal ideal domains only or can it be extended to a broader class of domains? We cannot definitely hope to extend it to the class of all domains since the structure of general domains could be very wild. Nevertheless, it turned out that the key property that Belkin used was the unique factorisation of principal ideals into a product of principal prime ideals. If we drop the word "principal" then we naturally come up with the notion of a Dedekind domain. Although it is true that the structure of general modules over Dedekind rings is not well described, we are actually interested in quasivarieties and they are always generated by finitely generated algebras. Hence what we really need are finitely generated Dedekind modules only and their structure is well understood.

The article has the following structure: Section 2 comes with basic notions and facts of ring theory. In Section 3 we define Dedekind rings and recall some of their basic properties. In Section 4 we recall properties of finitely generated modules over Dedekind rings and also Belkin's

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result. In Section 5 we focus on deductive varieties. Finally, Section 6 is the core of the paper with the main result, namely Theorem 6.9, describing the quasi-variety lattice of modules over Dedekind rings.

2. Basic definitions

In this paper we will assume that a ring $\mathcal{R}$ is commutative, with the identity element 1 and $0 \neq 1$. Let us recall some definitions from ring theory. *A domain* is a ring such that $ab = 0$ implies that either $a = 0$ or $b = 0$. For any ideals $\mathfrak{a}$, $\mathfrak{b}$ in a ring $\mathcal{R}$ we define *the product of ideals* $\mathfrak{ab}$ as follows:

$$\mathfrak{ab} = \left\{ \sum_{i=1}^{n} a_i b_i,\ a_i \in \mathfrak{a},\ b_i \in \mathfrak{b},\ n \geq 1 \right\}.$$

If $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ are ideals, then their product $\mathfrak{a}_1 \cdot \ldots \cdot \mathfrak{a}_n$ is analogously the set of all finite sums $\Sigma_i\ a_{1,i} \cdot \ldots \cdot a_{n,i}$ where $a_{k,i} \in \mathfrak{a}_k$ for $k = 1, \dots, n$.

The main tool in our study is, in fact, the divisibility lattice of ideals in a ring $\mathcal{R}$.

Definition 2.1. For ideals $\mathfrak{a}$ and $\mathfrak{b}$ in a commutative ring, write $\mathfrak{a} \mid \mathfrak{b}$ if $\mathfrak{b} = \mathfrak{ac}$ for some ideal $\mathfrak{c}$ and we say that the ideal $\mathfrak{a}$ divides the ideal $\mathfrak{b}$.

If $\mathfrak{a} \mid \mathfrak{b}$ then for some ideal $\mathfrak{c}$, we have $\mathfrak{b} = \mathfrak{ac}$ and $\mathfrak{ac} \subseteq \mathfrak{a}$, so $\mathfrak{a} \supseteq \mathfrak{b}$. Divisibility implies containment. The converse may fail in some rings, but in a Dedekind domain it will turn out that containment implies divisibility. So it is useful to think about containment of ideals in any ring as a preliminary form of divisibility: $\mathfrak{a} \supseteq \mathfrak{b}$ is, in general, something like $\mathfrak{a} \mid \mathfrak{b}$ and in our case, these two notions coincide.

Let $I(\mathcal{R})$ be the set of all ideals of the ring $\mathcal{R}$. The set $I(\mathcal{R})$ with the relation $\mid$ forms a partially ordered set which turns out to be a lattice:

$$\mathfrak{a} \wedge \mathfrak{b} = \mathfrak{a} \cap \mathfrak{b}, \quad \mathfrak{a} \vee \mathfrak{b} = \mathfrak{a} + \mathfrak{b},$$

where $\mathfrak{a} + \mathfrak{b}$ is the smallest ideal containing $\mathfrak{a} \cup \mathfrak{b}$.

Proposition 2.2. *The algebra* $(I(\mathcal{R}), \wedge, \vee)$ *is a complete modular lattice.*

In fact, in our case of Dedekind domains the lattice is distributive [11].

3. Dedekind rings

In this section we recall the definition and some main properties of Dedekind domains.

Definition 3.1. A proper ideal $\mathfrak{a}$ of a ring $\mathcal{R}$ is called:

(1) *prime* if $a \cdot b \in \mathfrak{a} \Rightarrow (a \in \mathfrak{a}$ or $b \in \mathfrak{a};)$
(2) *maximal* if for any ideal $\mathfrak{b}$ of the ring $\mathcal{R}$ if $\mathfrak{a} \subseteq \mathfrak{b} \subseteq \mathcal{R}$ then $\mathfrak{a} = \mathfrak{b}$ or $\mathfrak{b} = \mathcal{R}$.

The ring $\mathbb{Z}[\sqrt{-5}]$ is not a principal ideal domain since it is not a unique factorization domain. For example $9 = 3 \cdot 3 = (2 + \sqrt{-5}) \cdot (2 - \sqrt{-5})$. Each factor is irreducible and any two different elements are not associate. Then the principal ideal (9) is a product of principal ideals $(9) = (3) \cdot (3) = (2 + \sqrt{-5}) \cdot (2 - \sqrt{-5})$. Nevertheless, the ideal (9) is a unique product of prime ideals. Indeed, the ideals (3), $(2 + \sqrt{-5})$ and $(2 - \sqrt{-5})$ are not prime, but they are unique products of powers of prime ideals. Moreover, $(9) = (3, 2 + \sqrt{-5})^2 \cdot (3, 2 - \sqrt{-5})^2$ is the unique factorization.

We will consider domains, where the unique factorization of elements property is not required, but every nonzero proper ideal is a unique product of prime ideals.

Definition 3.2. A ring $\mathcal{R}$ is said to be *a Dedekind ring* if it is an integral domain and if every nonzero proper ideal of $\mathcal{R}$ is a product of prime ideals.

Theorem 3.3. *If $\mathcal{R}$ is a Dedekind ring then the product decomposition is unique, up to permutation.*

Lemma 3.4. *Let $\mathfrak{a}$ be an ideal of $\mathcal{R}$ and let $\mathfrak{p}$ be a prime ideal of $\mathcal{R}$. Then $\mathfrak{p}$ appears in the decomposition of $\mathfrak{a}$ if and only if $\mathfrak{a} \subseteq \mathfrak{p}$.*

Now we recall some basic properties of Dedekind rings. Let $\mathcal{R}$ be a integral domain and let $\mathcal{P}$ be a subring of $\mathcal{R}$.

Definition 3.5. We say that an element $x \in \mathcal{R}$ is *integral* over $\mathcal{P}$ if there exist a positive natural number n and elements $a_1, \ldots, a_n \in \mathcal{P}$ such that

$$x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0.$$

The set of all elements of the ring $\mathcal{R}$ integral over $\mathcal{P}$ is called *the integral closure* of $\mathcal{P}$ in $\mathcal{R}$ and is denoted by $C_{\mathcal{R}}(\mathcal{P})$.

We say that a ring $\mathcal{R}$ is *integrally closed* if $C_{\mathcal{F}}(\mathcal{R}) = \mathcal{R}$, where $\mathcal{F}$ is the fraction field of $\mathcal{R}$.

Theorem 3.6. *An integral domain $\mathcal{R}$ is a Dedekind ring if and only if the following conditions are satisfied:*

(DR1) *$\mathcal{R}$ is Noetherian;*
(DR2) *every non-zero prime ideal is maximal;*
(DR3) *R is integrally closed.*

Example 3.7. The following rings are Dedekind rings:

(1) A field,
(2) A principal ideal domain,
(3) The ring $\mathbb{Z}[\sqrt{-n}]$ where n is a square-free natural number and $n \equiv 1, 2 \pmod 4$.

Using the factorization of ideals, we can compute the greatest common divisor and the least common multiple of two nonzero ideals of a Dedekind domain.

Lemma 3.8. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a Dedeking ring $\mathcal{R}$. Then*

$$\mathrm{GCD}(\mathfrak{a}, \mathfrak{b}) = \mathfrak{a} + \mathfrak{b} = \mathfrak{a} \vee \mathfrak{b};$$
$$\mathrm{LCM}(\mathfrak{a}, \mathfrak{b}) = \mathfrak{a} \cap \mathfrak{b} = \mathfrak{a} \wedge \mathfrak{b}.$$

A Dedeking domain is not a principal ideal domain but it is close to it: every ideal has a two element generating set and, moreover, one of the generators can be chosen freely.

Proposition 3.9. *Let $\mathfrak{a}$ be a nonzero ideal of the Dedekind domain, and let r be any nonzero element of $\mathfrak{a}$. Then $\mathfrak{a}$ can be generated by two elements, one of which is r.*

Proof. Since $r \in \mathfrak{a}$ we have $(r) \subseteq \mathfrak{a}$. So $\mathfrak{a}$ divides (r), hence $(r) = \mathfrak{a}\mathfrak{b}$ for some ideal $\mathfrak{b}$. The ideal $\mathfrak{b}$ is a product $\mathfrak{p}_1^{\alpha_1} \dots \mathfrak{p}_n^{\alpha_n}$, where $\mathfrak{p}_i$ are distinct prime ideals. Let $\mathfrak{b}_i = \mathfrak{p}_1^{\alpha_1} \dots \mathfrak{p}_{i-1}^{\alpha_{i-1}} \mathfrak{p}_{i+1}^{\alpha_{i+1}} \dots \mathfrak{p}_n^{\alpha_n}$. Then $\mathfrak{a}\mathfrak{b}_i \supset \mathfrak{a}\mathfrak{b}$. Choose an element s_i such that $s_i \in \mathfrak{a}\mathfrak{b}_i$ and $s_i \notin \mathfrak{a}\mathfrak{b}$. Then take $s = s_1 + \dots + s_n$. We show that $s \in \mathfrak{a}$ and $s \notin \mathfrak{a}\mathfrak{p}_i^{\alpha_i}$ for all i. First, for all $s_i \in \mathfrak{a}\mathfrak{b}_i \subseteq \mathfrak{a}$, hence $s \in \mathfrak{a}$. Now s_i cannot belong to $\mathfrak{a}\mathfrak{p}_i^{\alpha_i}$, for if so, $s_i \in \mathfrak{a}\mathfrak{p}_\mathfrak{i}{}^{\alpha_i} \cap \mathfrak{a}\mathfrak{b}_i = LCM(\mathfrak{a}\mathfrak{p}_\mathfrak{i}{}^{\alpha_i}, \mathfrak{a}\mathfrak{b}_\mathfrak{i})$. But now last common multiple is $\mathfrak{a}\mathfrak{b}$. Hence $s_i \in \mathfrak{a}\mathfrak{b}$, a contradiction. We can write s as $s = (s_1 + \dots + s_{i-1}) + s_i + (s_{i+1} + \dots + s_n)$. For $i \neq j$ $s_j \in \mathfrak{a}\mathfrak{b}_j \subseteq \mathfrak{a}\mathfrak{p}_i^{\alpha_i}$. Hence the first and the third component of s are in $\mathfrak{a}\mathfrak{p}_i^{\alpha_i}$ and the second is not. Hence $s \notin \mathfrak{a}\mathfrak{p}_i^{\alpha_i}$.

As $s \in \mathfrak{a}$, we have $(s) \subseteq \mathfrak{a}$ and there exists an ideal $\mathfrak{a}'$ such that $(s) = \mathfrak{a}\mathfrak{a}'$. If any $\mathfrak{p}_i^{\alpha_i}$ divides $\mathfrak{a}'$ then $\mathfrak{a}' = \mathfrak{p}_i^{\alpha_i}\mathfrak{a}''$ and $(s) = \mathfrak{a}\mathfrak{p}_i^{\alpha_i}\mathfrak{a}''$ hence $s \in \mathfrak{a}\mathfrak{p}_i^{\alpha_i}$ a contradiction. Finally we obtain that $\mathfrak{a}'$ is relatively prime to $\mathfrak{b}$ and $GCD((r), (s)) = GCD(\mathfrak{a}\mathfrak{b}, \mathfrak{a}\mathfrak{a}') = \mathfrak{a}$, as $\mathfrak{a}$ and $\mathfrak{a}'$ are relatively prime. Then $\mathfrak{a} = (r, s)$.

$\square$

Proposition 3.10. *If a Dedekind domain $\mathcal{R}$ has only finitely many prime ideals, then it is a principal ideal domain.*

4. Modules over Dedekind rings

In this section we recall some properties of finitely generated modules over a Dedekind domain. All rings considered here are thus Dedekind domains.

Definition 4.1. Let $\mathcal{M}$ be an R-module. If x is any element of $\mathcal{M}$, then the set of all r in $\mathcal{R}$ with $rx = 0$ is called the *annihilator* of x. If all elements of $\mathcal{M}$ have a nonzero annihilator, we call $\mathcal{M}$ a *torsion module.* If all non-zero elements of $\mathcal{M}$ have $\{0\}$ as the annihilator then $\mathcal{M}$ is called a *torsion free module.*

Theorem 4.2. [9, Theorem 1.41] *Let $\mathcal{M}$ be a finitely generated non-trivial torsion $\mathcal{R}$-module. Then there exist prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ in $\mathcal{R}$ and positive natural numbers k_i for $i = 1, \ldots, n$, such that $\mathcal{M}$ is isomorphic to the sum*

$$\mathcal{M} \cong \mathcal{R}/\mathfrak{p}_1^{k_1} \oplus \ldots \oplus \mathcal{R}/\mathfrak{p}_n^{k_n}.$$

Corollary 4.3. *Let $\mathcal{M}$ be a finitely generated non-trivial torsion module. Then every homomorphic image of $\mathcal{M}$ embeds into $\mathcal{M}$.*

Lemma 4.4. [9, Lemma 1.38] *For every domain R any finitely generated and torsion-free R-module M is a submodule of a free R-module.*

Lemma 4.5. *Let $\mathcal{M}$ be a finitely generated non-trivial module over $\mathcal{R}$ and let $\mathfrak{p}$ be a prime ideal of $\mathcal{R}$. We define $\mathcal{M}_\mathfrak{p} = \{x \in \mathcal{M},\ \mathfrak{p}^m(x) = (0),$ for some $m\}$. Then there exist $k_1 \leq k_2 \leq \ldots \leq k_n$ such that*

$$\mathcal{M}_\mathfrak{p} \cong \mathcal{R}/\mathfrak{p}^{k_1} \oplus \ldots \oplus \mathcal{R}/\mathfrak{p}^{k_n}.$$

The structure of a finitely generated non-trivial $\mathcal{R}$-module $\mathcal{M}$ is presented in the following theorem:

Theorem 4.6. [9, Theorem 1.32] *Let $\mathcal{M}$ be a finitely generated non-trivial $\mathcal{R}$-module and $\mathcal{M}_T$ be its submodule consisting of all torsion elements, i.e., of all elements $x \in \mathcal{M}$, which, for some non-zero $r \in \mathcal{R}$, satisfy $rx = 0$. Then $\mathcal{M}$ is isomorphic to a direct sum*

$$\mathcal{M} \cong \mathcal{R}^n \oplus \mathfrak{a} \oplus \mathcal{M}_T,$$

where $n \in \mathbb{N}$ and $\mathfrak{a}$ is an ideal of $\mathcal{R}$.

In a variety of modules over PID every ideal, as an R-module, is isomorphic to the free R-module R. This is not true for Dedekind rings in general.

Example 4.7. In the variety of $\mathbb{Z}[\sqrt{-5}]$-modules the following modules are not isomorphic

$$(3, 2 - \sqrt{-5}) \ncong (3, 2 + \sqrt{-5}).$$

For a Dedekind domain $\mathcal{R}$ which is not a PID, there are infinitely many non-isomorphic ideals and therefore infinitely many non-isomorphic and torsion-free $\mathcal{R}$-modules.

Lemma 4.8. [9, Lemma 1.38] *For every domain $\mathcal{R}$ any finitely generated and torsion-free $\mathcal{R}$-module $\mathcal{M}$ is a submodule of a free $\mathcal{R}$-module.*

The lattice of quasivarieties of modules over a principal ideal domain was described in [1] as follows.

Let $\mathcal{R}$ be a principal ideal domain and $\mathbb{P}(\mathcal{R})$ be the set of all prime elements of the ring $\mathcal{R}$. The lattice $\mathcal{L}_q(\mathbf{Mod}_{\mathcal{R}})$ of subquasivarieties of $\mathbf{Mod}_{\mathcal{R}}$ over $\mathcal{R}$ the principal ideals domain, may be characterized using the lattice $\mathcal{L}(\alpha)$ introduced by Belkin [1], and defined as follows: (recall that a cardinal number is the least ordinal of the given cardinality and also that an ordinal number is defined as the set of all smaller ordinal numbers). For each cardinal number α, $i \in \alpha$ is an ordinal number.

Definition 4.9. [1] Let α^+ denote the union $\alpha \cup \{\infty\}$. Let $\mathcal{L}(\alpha)$ be the set of functions

$$f : \alpha^+ \to \omega^+,$$

satisfying $f(\infty) \in \{0, \infty\}$, with $f(\infty) = 0$ only if $f(\alpha) \not\ni \infty$ and $f(i) = 0$, for almost all $i \in \alpha$. Then $\mathcal{L}(\alpha)$ is a distributive lattice with respect to the following operations:

$$(f \vee g)(i) = \max\{f(i), g(i)\}, \ (f \wedge g)(i) = \min\{f(i), g(i)\},$$

where $i < \infty$ for all $i \in \alpha$.

Theorem 4.10. [1, Theorem 2.1] *Let the ring $\mathcal{R}$ be a principal ideal domain, and $|\mathbb{P}| = \alpha$, where $\mathbb{P}$ is the set of prime elements in the ring $\mathcal{R}$. Then the lattice of quasivarieties of the variety of modules over the ring $\mathcal{R}$ is isomorphic to the lattice $\mathcal{L}(\alpha)$, i.e.,*

$$\mathcal{L}_q(\mathbf{Mod}_{\mathcal{R}}) \cong \mathcal{L}(\alpha).$$

5. Deductive subvarieties of the variety of modules over a Dedekind ring

In Theorem 4.10, a part of the quasivarieties lattice consists of varieties only, namely, the quasivarieties generated by finite modules. The same happens in the Dedekind case, as we shall see in this section. What are finite Dedekind modules? The variety corresponding to an ideal $\mathfrak{a}$ is generated by the module $\mathcal{R}/\mathfrak{a}$ and denoted by $\mathcal{V}_{\mathfrak{a}}$. As any ideal $\mathfrak{a}$ of a Dedekind domain has two generators r and p, it follows that the subvariety corresponding to this ideal is defined by two identities $px = 0$ and $rx = 0$.

The following result is well-known.

Theorem 5.1. *Let $\mathcal{R}$ be an arbitrary ring. The lattice of varieties $\mathcal{L}(\mathbf{Mod}_{\mathcal{R}})$ of the variety $\mathbf{Mod}_{\mathcal{R}}$ of modules over the ring $\mathcal{R}$ is dually isomorphic to the lattice of ideals of $\mathcal{R}$.*

Definition 5.2. We say that a variety $\mathcal{V}$ is *deductive* if each subquasivariety of $\mathcal{V}$ is a variety.

We want to prove that every proper subvariety of $\mathbf{Mod}_{\mathcal{R}}$ where $\mathcal{R}$ is a Dedekind domain, is deductive. It is not difficult to prove this result directly. However, it is easier to use a characterization of deductive varieties provided by L. Hogben and C. Bergman [4].

Definition 5.3. An algebra $\mathrm{P} \in \mathcal{V}$ is *primitive* if P is finite, subdirectly irreducible and, for all $\mathrm{A} \in \mathcal{V}$, $\mathrm{P} \in H(\mathrm{A}) \Rightarrow \mathrm{P} \in IS(\mathrm{A})$.

Theorem 5.4. [4, Theorem 3.4] *Let $\mathcal{V}$ be residually finite and of finite type, or residually and locally finite. Then $\mathcal{V}$ is deductive if and only if every subdirectly irreducible algebra in $\mathcal{V}_{SI}$ is primitive.*

Corollary 5.5. *Let $\mathcal{R}$ be a Dedekind domain. Each proper subvariety of the variety $\mathbf{Mod}_{\mathcal{R}}$ is deductive.*

Proof. Every proper subvariety of $\mathcal{R}$ is $\mathcal{V}_{\mathfrak{a}}$, for some ideal $\mathfrak{a}$. This variety is locally finite. The subdirectly irreducible members of the variety $\mathcal{V}_{\mathfrak{a}}$ are $\mathcal{R}/\mathfrak{p}^k$, for some natural k such that $\mathfrak{p}^k|\mathfrak{a}$ and $\mathfrak{p}$ is a prime ideal, and hence $\mathcal{V}_{\mathfrak{a}}$ is residually finite. According to Corollary 4.3, all the homomorphic images of torsion modules are submodules and hence all the subdirectly irreducibles are primitive. $\square$

Example 5.6. Let $\mathfrak{p}_1, \mathfrak{p}_2$ be prime ideals, then the lattice of subquasivarieties $\mathcal{L}_q(\mathcal{V}_{\mathfrak{p}_1\mathfrak{p}_2})$ consists of the four members displayed in Figure 1

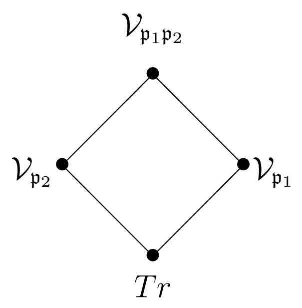

FIGURE 1. Subquasivarieties of $\mathcal{V}_{\mathfrak{p}_1\mathfrak{p}_2}$

Theorem 5.7. *Let $\mathcal{R}$ be a Dedekind domain and let $\mathfrak{a}$ be an ideal of this ring. The lattice $\mathcal{L}_q(\mathcal{V}_{\mathfrak{a}}) = \mathcal{L}(\mathcal{V}_{\mathfrak{a}})$ is isomorphic to the lattice of divisors of $\mathfrak{a}$ under divisibility.*

6. The lattice of subquasivarieties of the variety of modules over a Dedekind ring

In this section we show that the lattice of quasivarieties of modules over a Dedekind domain $\mathcal{R}$ depends on the number of prime ideals of the ring $\mathcal{R}$.

Let $\mathcal{R}$ be a Dedekind domain.

Lemma 6.1. *For every Dedekind domain $\mathcal{R}$ and a finitely generated torsion-free $\mathcal{R}$-module $\mathcal{M} = \mathcal{R}^n \oplus \mathfrak{a}$, where $\mathfrak{a}$ is an ideal of $\mathcal{R}$, we have:*

$$\mathcal{Q}(\mathcal{R}^n \oplus \mathfrak{a}) = \mathcal{Q}(\mathcal{R}).$$

Proof. According to Lemma 4.4, any finitely generated and torsion-free R-module $\mathfrak{a}$ is a submodule of a free R-module and therefore the following inclusion holds:

$$\mathcal{Q}(\mathcal{R}^n \oplus \mathfrak{a}) \subseteq \mathcal{Q}(\mathcal{R} \oplus \mathfrak{a}) \subseteq \mathcal{Q}(\mathcal{R}).$$

Let $a, b \in \mathcal{R}$ be generators of the $\mathcal{R}$-module $\mathfrak{a} = (a, b)$. Then $\mathfrak{a}_1 = (a)$ is a submodule of the $\mathcal{R}$-module $\mathfrak{a}$ and the module $\mathfrak{a}_1$ is isomorphic to the free $\mathcal{R}$-module $\mathcal{R}$. Therefore:

$$\mathcal{Q}(\mathcal{R}^n \oplus \mathfrak{a}) \supseteq \mathcal{Q}(\mathcal{R}^n \oplus \mathfrak{a}_1) \supseteq \mathcal{Q}(\mathcal{R}^{n+1}) = \mathcal{Q}(\mathcal{R}).$$

Both inclusions hold. □

Lemma 6.2. *The quasivariety $\mathcal{Q}(\mathcal{R})$ generated by the $\mathcal{R}$-module $\mathcal{R}$ is the only minimal quasivariety which is not a variety. The $\mathcal{R}$-module $\mathcal{R}$ is relatively subdirectly irreducible in the quasivariety $\mathcal{Q}(\mathcal{R})$.*

Proof. The only non-trivial submodules of $\mathcal{R}$ are non-trivial ideals of $\mathcal{R}$ which contain submodules isomorphic to $\mathcal{R}$. Hence $\mathcal{Q}(\mathcal{R})$ is minimal. All quasivarieties contain either $\mathcal{R}$ or $\mathfrak{a}$ or some quotient of $\mathcal{R}$. □

If a Dedekind domain $\mathcal{R}$ is a PID then the $\mathcal{R}$-module $\mathcal{R}$ is the only relatively subdirectely irreducible in the quasivariety $\mathcal{Q}(\mathcal{R})$. If $\mathcal{R}$ is a Dedekind domain which is not a PID then the quasivariety $\mathcal{Q}(\mathcal{R})$ contains infinitely many non-isomorphic relatively subdirectly irreducible modules.

Lemma 6.3. *Each $\mathcal{R}$-module $\mathfrak{a}$, for $\mathfrak{a}$ an ideal of $\mathcal{R}$, is relatively subdirectly irreducible in the quasivariety $\mathcal{Q}(\mathcal{R}, \mathcal{R}/\mathfrak{a}_1, \dots, \mathcal{R}/\mathfrak{a}_n)$ for some ideals $\mathfrak{a}_i$ for $i = 1, \dots, n$, $n \geq 0$. On the other hand, every finitely generated relatively subdirectly irreducible $\mathcal{R}$-module in this quasivariety is either finite subdirectly irreducible or isomorphic to an ideal $\mathfrak{a}$.*

Proof. The $\mathcal{Q}$-congruence lattice of a $\mathcal{R}$-module $\mathfrak{a}$ has the monolith: the smallest non-trivial $\mathcal{Q}$-ideal is $\mathfrak{a}(\mathfrak{a}_1 \cap \mathfrak{a}_2 \cap \dots \cap \mathfrak{a}_n)$. On the other hand, every finitely generated $\mathcal{R}$-module is isomorphic to $\mathcal{R}^n \oplus \mathfrak{a} \oplus \mathcal{M}_T$, according to Theorem 4.6. Hence every relatively subdirectly irreducible is either isomorphic to $\mathfrak{a}$ or finite torsion module. And a quasivariety generated by a finite module is a variety hence all relatively subdirectly irreducible are subdirectly irreducible. □

Lemma 6.4. *The $\mathcal{R}$-module $\mathcal{R}$ belongs to any quasivariety $\mathcal{Q}$, containing $\mathcal{R}/\mathfrak{a}_1, \mathcal{R}/\mathfrak{a}_2, \ldots$, for infinitely many pairwise different ideals $\mathfrak{a}_i$, for $i \in \mathbb{N}$. Moreover, the $\mathcal{R}$-module $\mathcal{R}$ is not subdirectly irreducible relatively to $\mathcal{Q}$.*

Proof. The ideal $\mathfrak{a}_1 \cap \mathfrak{a}_2 \cap \cdots$ is trivial and hence the $\mathcal{Q}$-congruence lattice of the $\mathcal{R}$-module $\mathcal{R}$ does not have a monolith. And this means that we have $\mathcal{R} \leq \prod \mathcal{R}/\mathfrak{a}_i$. □

Recall that each ideal in a Dedekind domain is generated by two elements. In the sequel we will define quasi-identities that distinguish specific prime ideals. In particular, we need to measure the valuation of prime-ideals, optimally by a single element for each prime ideal. If the prime ideal is principal then, naturally, we use the generator. If it is not principal then an arbitrary generator may not do the job, as we see in the next example:

Example 6.5. Let $\mathcal{R} = \mathbb{Z}[\sqrt{-5}]$. The prime ideal $(3; 2+\sqrt{-5})$ can be represented by many different pairs of generators, e.g. $(9; 5+\sqrt{-5})$. But here the number 9 is not good for our purposes since $9 \in (9; 5+\sqrt{-5})^2 = (3; 2+\sqrt{-5})^2$.

Lemma 6.6. *Let $\mathfrak{p}$ be a prime ideal of $\mathcal{R}$. Let $a \in \mathfrak{p} \smallsetminus \mathfrak{p}^2$. Then $a^k \in \mathfrak{p}^k \smallsetminus \mathfrak{p}^{k+1}$, for each $k \in \mathbb{N}$.*

Proof. According to Lemma 3.4, the decomposition of (a) is $(a) = \mathfrak{p} \cdot \prod \mathfrak{q}_i^{n_i}$, for some prime ideals $\mathfrak{q}_i$ distinct from $\mathfrak{p}$ and exponents n_i, since $a \in \mathfrak{p}$ and $a \notin \mathfrak{p}^2$. Now $(a^k) = (a)^k = \mathfrak{p}^k \cdot \prod \mathfrak{q}_i^{kn_i}$, showing the claim. □

For the rest of the section we shall use the following notation: fix $\mathfrak{a}$, an ideal of $\mathcal{R}$ and let $\mathfrak{a} = \mathfrak{p}_{i_1}^{k_1} \ldots \mathfrak{p}_{i_n}^{k_n}$, where $\mathfrak{p}_{i_1}, \ldots, \mathfrak{p}_{i_n}$ are prime ideals. Each prime ideal $\mathfrak{p}_{i_j}$, for $i \in \{1, 2, \ldots, n\}$, has two generators. We write

$$\mathfrak{p}_{i_j} = (p_{i_j,1};\ p_{i_j,2})$$

(if $\mathfrak{p}_{i_j}$ is principal then $p_{i_j,2}$ can be arbitrary, e.g. 0) and we always choose $p_{i_j,1} \in \mathfrak{p}_{i_j} \smallsetminus \mathfrak{p}_{i_j}^2$. Such an element always exists since $\mathfrak{p}_{i_j} \supsetneq \mathfrak{p}_{i_j}^2$ due to the uniqueness of the decomposition.

Now, since $p_{i_j,1}^{k_j}$ lies in $\mathfrak{p}_{i_j}^{k_j}$, there exists an element $p_{i_j,k_j,2} \in \mathfrak{p}_{i_j}^{k_j}$, such that

$$\mathfrak{p}_{i_j}^{k_j} = (p_{i_j,1}^{k_j};\ p_{i_j,k_j,2})$$

and analogously there exists an element $p_{(i_j),(k_j),2}$, such that

$$\mathfrak{a} = \mathfrak{p}_{i_1}^{k_1} \dots \mathfrak{p}_{i_n}^{k_n} = \left(\prod_{j \in \{1,\dots,n\}} p_{i_j,1}^{k_j};\ p_{(i_j),(k_j),2} \right).$$

The subvariety $\mathcal{V}(\mathcal{R}/\mathfrak{a})$ of $\mathbf{Mod}_{\mathcal{R}}$ is defined by only two identities:

$$Mod \left\{ \prod_{j \in \{1,\dots,n\}} p_{i_j,1}^{k_j} x = 0;\ p_{(i_j),(k_j),2} x = 0 \right\} = \mathcal{V}(\mathcal{R}/\mathfrak{a})$$

and by the Chinese remainder theorem

$$\mathcal{V}(\mathcal{R}/\mathfrak{a}) = \mathcal{V}(\prod_{j \in \{1,\dots,n\}} \mathcal{R}/\mathfrak{p}_{i_j}^{k_j}) = \mathcal{V}(\mathcal{R}/\mathfrak{p}_{i_1}^{k_1}, \mathcal{R}/\mathfrak{p}_{i_2}^{k_2}, \dots, \mathcal{R}/\mathfrak{p}_{i_n}^{k_n}).$$

Lemma 6.7. *Let $\mathcal{Q}$ be the subquasivariety of the variety $\mathbf{Mod}_{\mathcal{R}}$ and let $\mathfrak{p}_i^{k+1} = (p_{i,1}^{k+1}, p_{i,k+1,2})$ be the $k+1$-th power of a prime ideal $\mathfrak{p}_i = (p_{i,1}, p_{i,2})$ of the ring $\mathcal{R}$, for some $k \in \mathbb{N}$. If*

$$\mathcal{Q} \models (p_{i,1}^{k+1} x = 0 \ \ \&\ \ p_{i,k+1,2} x = 0 \ \rightarrow\ p_{i,1}^{k} x = 0),$$

then the module $\mathcal{R}/\mathfrak{p}_i^{k+1}$ does not belong to the quasivariety $\mathcal{Q}$ and $\mathcal{R}/\mathfrak{p}_i^{k}$ belongs to the quasivariety $\mathcal{Q}$.

Proof. Taking the element $1+\mathfrak{p}^{k+1} \in \mathcal{R}/\mathfrak{p}_i^{k+1}$, we have $p_{i,1}^{k+1}(1+\mathfrak{p}^{k+1}) = 0 + \mathfrak{p}^{k+1}$ since $p_{i,1}^{k+1} \in \mathfrak{p}^{k+1}$ and $p_{i,1}^{k}(1+\mathfrak{p}^{k+1}) \neq 0 + \mathfrak{p}^{k+1}$, according to Lemma 6.6. Moreover $p_{i,k+1,2}(1+\mathfrak{p}_i^{k+1}) = 0 + \mathfrak{p}_i^{k+1}$ and therefore the element $1+\mathfrak{p}_i^{k+1}$ satisfies the premises of the quasi-identity and does not satisfy the conclusion. Hence

$$\mathcal{R}/\mathfrak{p}_i^{k+1} \not\models (p_{i,1}^{k+1} x = 0 \ \&\ p_{i,k+1,2} x = 0 \rightarrow p_{i,1}^{k} x = 0).$$

On the other hand,

$$\mathcal{R}/\mathfrak{p}_i^{k} \models (p_{i,1}^{k+1} x = 0 \ \&\ p_{i,k+1,2} x = 0 \rightarrow p_{i,1}^{k} x = 0),$$

because each element of the module $\mathcal{R}/\mathfrak{p}_i^{k}$ satisfies the conclusion of the quasi-identity. □

Remark 6.8. If $k = 0$ in the previous lemma then the quasi-identity is of the form:

$$p_{i,1} x = 0 \ \&\ p_{i,2} x = 0 \rightarrow x = 0$$

and each element of the module $\mathcal{R}/\mathfrak{p}_i$ satisfies the premises of the quasi-identity. However, only 0 satisfies the conclusion.

Let $|\mathbb{P}(\mathcal{R})| = \alpha$ be the cardinality of the set of the prime ideals in the ring $\mathcal{R}$. (Recall that a cardinal number is the least ordinal of the given cardinality). Suppose that the set $\mathbb{P}(\mathcal{R})$ is well ordered, i.e., the elements of $\mathbb{P}(\mathcal{R})$ are indexed by the elements of α. As in the case of PID, the lattice $\mathcal{L}_q(\mathbf{Mod}_{\mathcal{R}})$ is isomorphic to a lattice defined in Definition 4.9.

Theorem 6.9. *Let $\mathcal{R}$ be a Dedekind ring and let $|\mathbb{P}(\mathcal{R})| = \alpha$ be the cardinality of the set of the prime ideals in the ring $\mathcal{R}$. Then the lattice of quasivarieties of the variety of modules over the Dedekind ring $\mathcal{R}$ is isomorphic to the lattice $\mathcal{L}(\alpha)$*

$$\mathcal{L}_q(\mathbf{Mod}_{\mathcal{R}}) \cong \mathcal{L}(\alpha).$$

The isomorphism $\varphi : \mathcal{L}(\alpha) \to \mathcal{L}_q(\mathbf{Mod}_{\mathcal{R}})$ is defined as follows:

$$\varphi(f) = Mod\Sigma_f,$$

where Σ_f is the set of quasi-identities:

(a) *if $f(\infty) = \infty$, then Σ_f contains*

$$(p_{i,1}^{f(i)+1}x = 0 \ \& \ p_{i,f(i)+1,2}x = 0 \to p_{i,1}^{f(i)}x = 0),$$

$$\text{for } \mathfrak{p}_i \in \mathbb{P}(\mathcal{R}), \text{ whenever } f(i) \neq \infty, \quad (\beta_{\mathfrak{p}_i^{f(i)}})$$

(b) *if $f(\infty) = 0$, then Σ_f contains only the identities:*

$$\prod_{f(i)\neq 0} p_{i,1}^{f(i)}x = 0 \ \& \ p_{(i),(f(i)),2}x = 0, \qquad (\gamma_{\mathfrak{p}_i^{f(i)}})$$

where $(\prod_{f(i)\neq 0} p_{i,1}^{f(i)}, p_{(i),(f(i)),2})$ is the generating pair of the ideal $\mathcal{I} = \mathfrak{p}_{i_1}^{f(i_1)} \cdot \ldots \cdot \mathfrak{p}_{i_n}^{f(i_n)}$ for each $0 \neq f(i_j) < \infty$, $j \in \{1, \ldots, n\}$.

Proof. We show first that the function φ is surjective. Let $\mathcal{Q}$ be a subquasivariety of $\mathbf{Mod}_{\mathcal{R}}$. Let us define a function $f : \alpha^+ \to \omega \cup^+$ as follows

$$f(\infty) = \begin{cases} \infty & if \quad \mathcal{R} \in \mathcal{Q} \\ 0 & if \quad \mathcal{R} \notin \mathcal{Q} \end{cases}$$

and for an ordinal number $i \in \alpha$, $f(i) = \sup\{k : \ \mathcal{R}/\mathfrak{p}_i^k \in \mathcal{Q})\}$ for $i \neq \infty$. The function f is well defined: we see $f(\infty) \in \{0, \infty\}$ we prove that $f(\infty) = 0$ implies $f(i) < \infty$, for all i, and $f(i) = 0$, for almost all $i \in \alpha$. Suppose first, by contradiction, that $f(\infty) = 0$ and $f(i) = \infty$ for some $i \in \alpha$. Then, according to Lemma 6.4, we obtain $\mathcal{R} \in \mathcal{Q}$ which contradicts $f(\infty) = 0$. Suppose now, that the number of elements of the set $\mathcal{I} = \{i : f(i) \neq 0\}$ is infinite. Then, according to Lemma 6.4 again, $\mathcal{R} \in \mathcal{Q}$, a contradiction.

We show now that $Mod\Sigma_f = \varphi(f) = \mathcal{Q}$. Consider two cases:

(a) $f(\infty)=\infty$: Let the module $\mathcal{M}\in\mathcal{Q}$, $f(i)\neq\infty$ and let there exists an element $m\in\mathcal{M}$ such that $p_i^{f(i)+1}m=0$ & $p_{i,f(i)+1,2}m=0$. Then the premises of the quasi-identity $(\beta_{\mathfrak{p}_i^{f(i)}})$ hold and the ideal (m) is a finitely generated $\mathcal{R}$-module and therefore we can use Lemma 4.5. We obtain

$$(m)=(m)_{\mathfrak{p}_i}\cong\mathcal{R}/\mathfrak{p}_i^{k_1}\oplus\ldots\oplus\mathcal{R}/\mathfrak{p}_i^{k_n},$$

for some $k_1\leq k_2\leq\ldots\leq k_n$. Since, by definition, $k_n\leq f(i)$, according to Lemma 6.6, we obtain $\mathfrak{p}_i^{f(i)}m=0$, $\mathcal{M}\models\beta_{\mathfrak{p}_i^{f(i)}}$ and $\mathcal{Q}\subseteq Mod\Sigma_f$.

On the other hand, let $\mathcal{M}$ be a generator of $Mod\Sigma_f$. Since a quasivariety is generated by finitely generated modules, we can assume $\mathcal{M}$ to be finitely generated. Then, according to Theorem 4.6,

$$\mathcal{M}\cong\mathcal{R}^n\oplus\mathfrak{a}\oplus\mathcal{M}_T.$$

The torsion-free part, that means $\mathcal{R}^n\oplus\mathfrak{a}$, belongs to $\mathcal{Q}$, according to Lemma 6.1. Now $\mathcal{M}_T\equiv\bigoplus_i\bigoplus_{k_{i,j}}\mathcal{R}/\mathfrak{p}_i^{k_{i,j}}$ and, according to Lemma 6.6, $k_{i,j}\leq f(i)$, for all i,j. Now the definition of the function f yields $\mathcal{M}_T\in\mathcal{Q}$ and therefore $Mod\Sigma_f\subseteq\mathcal{Q}$.

(b) $f(\infty)=0$: The module $\mathcal{R}\notin\mathcal{Q}$ and $f(\alpha)<\infty$ and $f(i)=0$, for almost all $i\in\alpha$. Let $\mathcal{M}\in\mathcal{Q}$, be finitely generated. Then $\mathcal{M}$ is torsion-free (otherwise $\mathcal{R}$ embeds in $\mathcal{M}$) and, according to Theorem 4.2, $\mathcal{M}$ is the sum of the modules $\mathcal{R}/\mathfrak{p}_i^{k_i}$ where $f(i)\neq 0$ and $k_i\leq f(i)$. Then

$$\mathcal{M}\models\left(\prod_{f(i)\neq 0}p_i^{f(i)}x=0\ \&\ p_{(i),(f(i)),2}x=0\right).$$

Hence $\mathcal{M}\in Mod\Sigma_f$.

On the other hand, we define $\mathfrak{a}=\cap_{i<\infty}\mathfrak{p}_i^{f(i)}$. Clearly $Mod\Sigma_f=\mathcal{V}(\mathcal{R}/\mathfrak{a})=\mathcal{V}(\bigoplus\mathcal{R}/\mathfrak{p}_i^{f(i)})\subseteq\mathcal{Q}$.

Finally, we prove injectivity. Let $f(i)\neq g(i)$ for some $f,g\in\mathcal{L}(\alpha)$. If $f(i)<g(i)$, then $\mathcal{R}/\mathfrak{p}_i^{g(i)}\in Mod\Sigma_g$ and $\mathcal{R}/\mathfrak{p}_i^{g(i)}\notin Mod\Sigma_f$. Then $\varphi(f)\neq\varphi(g)$ and $Mod\Sigma_f<Mod\Sigma_g$, $\varphi(f)<\varphi(g)$.

Similarly, we can show that φ preserves the lattices order. □

Example 6.10. Let $\mathcal{R}=\mathbb{Z}[\sqrt{-p}]$, then $|\mathbb{P}(\mathbb{Z}[\sqrt{-p}])|=\omega$ and

$$\mathcal{L}_q(\mathbf{Mod}_{\mathbb{Z}[\sqrt{-p}]})\cong\mathcal{L}(\omega).$$

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