EXAMPLES TO BIRKHOFF'S QUASIGROUP AXIOMS

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Abstract. The equational variety of quasigroups is defined by six identities, called Birkhoff's identities. It is known, that only four of them suffice to define the variety; actually, there are nine different combinations of four Birkhoff's identities defining quasigroups, other four combinations define larger varieties and it was open whether the remaining two cases define quasigroups or larger classes. We solve the question here constructing examples of algebras that are not quasigroups and satify the open cases of Birkhoff's identities.

1. Introduction

Quasigroups are binary systems $(Q, *)$ such that all equations

$$a * x = b \qquad \text{and} \qquad x * a = b, \qquad \text{for } a, b \in Q$$

have unique solutions. This natural definition, however, has a drawback that subalgebras do need to conserve the existence and homomorphic images the uniqueness of a solution. To deal with this problem, Birkhoff [2] added two more operations $/$ and $\backslash$ and six axioms

$$\begin{array}{llllll} x * (x\backslash y) = y & (1) & (y/x) * x = y & (2) & x\backslash(x * y) = y & (3) \\ (y * x)/x = y & (4) & x/(y\backslash x) = y & (5) & (x/y)\backslash x = y & (6). \end{array}$$

that are true in every quasigroup. Now quasigroups, as a class of algebras in signature $(*, /, \backslash)$, form a variety, that means a class closed for subalgebras, homomorphic images and products.

It turned out quite soon [4] that not all of the six identities are needed, that (1)–(4) suffice since (5) and (6) are consequences of (1)–(4). And it is a natural question: "Which other four-tuples of Birkhoff's identities do also define the entire equational class of quasigroup identities?" Phillips, Pushkashu, Shcherbacov and Shcherbacov [5] studied this question and proved that, among all fifteen four-tuples, there are nine of them defining quasigroups, one defining cancellative left quasigroups, one defining cancellative right quasigroups, one defining divisible left quasigroups and one defining divisible right quasigroups. The form of the equational variety of the remaining two combinations was not discovered and two open problems were thus formulated:

Problem 1. [5] *Is a binary algebra* $(Q, *, /, \backslash)$ *satisfying* (1), (2), (5), (6) *necessarily a quasigroup? Is a binary algebra* $(Q, *, /, \backslash)$ *satisfying* (3), (4), (5), (6) *necessarily a quasigroup?*

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The answer is negative to both questions, as we see using the following counterexamples:

Example 2. Consider the algebra $(\mathbb{Z}, *, /, \backslash)$ with operations

$$a * b = \left\lfloor \frac{b-a}{2} \right\rfloor, \qquad a/b = b - 2a, \qquad a\backslash b = a + 2b.$$

This algebra satisfies (1), (2), (5), (6) but not (3) and (4).

Example 3. Consider the algebra $(\mathbb{Z}, *, /, \backslash)$ with operations

$$a * b = 2(b-a), \qquad a/b = b - \lfloor a/2 \rfloor, \qquad a\backslash b = a + \lfloor b/2 \rfloor.$$

This algebra satisfies (3), (4), (5), (6) but not (1) and (2).

It is easy to check by hand that both examples have announced properties and therefore we could easily finish our paper here. Nevertheless, it might be interesting to show how the examples were constructed and this is the content of the second section.

2. Parastrophes

The key notion of our construction is parastrophy. Parastrophies are usually constructed for quasigroups only but here it is useful to generalize the notion for any algebras with three binary operations.

Definition 4. *Let $(Q, *, /, \backslash)$ be a binary algebra. We define* parastrophes $Q_i = (Q, *_i, /_i, \backslash_i)$*, for $i = 1, 2, 3, 4, 5$, of Q as*

$$\begin{aligned}
x *_1 y &= x\backslash y, & x/_1 y &= y/x, & x\backslash_1 y &= x * y;\\
x *_2 y &= x/y, & x/_2 y &= x * y, & x\backslash_2 y &= y\backslash x;\\
x *_3 y &= y/x, & x/_3 y &= x\backslash y, & x\backslash_3 y &= y * x;\\
x *_4 y &= y\backslash x, & x/_4 y &= y * x, & x\backslash_4 y &= x/y;\\
x *_5 y &= y * x, & x/_5 y &= y\backslash x, & x\backslash_5 y &= y/x;
\end{aligned}$$

If $(Q, *, /, \backslash)$ turns out to be a quasigroup then the parastrophes so defined are exactly the classical parastrophes with the notation taken from [3].

Recall that a *left translation* of an operation $\circ$ is a mapping L_a° that sends x to $a \circ x$. Analogously $R_a^\circ : x \mapsto x \circ a$. If $(Q, *, /, \backslash)$ is a quasigroup then all the left and right translations are bijections: Identities (1)–(6) can be rewritten as

$$\begin{aligned}
L_x^* L_x^\backslash &= \mathrm{id} \quad (1) & R_x^* R_x^/ &= \mathrm{id} \quad (2) & L_x^\backslash L_x^* &= \mathrm{id} \quad (3)\\
R_x^/ R_x^* &= \mathrm{id} \quad (4) & L_x^/ R_x^\backslash &= \mathrm{id} \quad (5) & R_x^\backslash L_x^/ &= \mathrm{id} \quad (6)
\end{aligned}$$

If we consider a parastrophe of an algebra, then, necessarily, translations of the parastrophe are the same as the translations of the original algebra, only the associated operations differ. Moreover, it turns out that a Birkhoff's identity is transformed into another Birkhoff's identity. In Table 1 we see what are images of the translations and of Birkhoff's identities under parastrophies.

There are examples of algebras that satisfy four Birkhoff's identities and not the other two. Solutions to Problem 1 are constructed as parastrophes of such well known examples.

Problem 5. *Construct an algebra satisfying* (1), (2), (5), (6) *and not* (3), (4).

Q	Q_1	Q_2	Q_3	Q_4	Q_5
L^*_a	$L^\backslash_a$	$L^/_a$	$R^\backslash_a$	$R^/_a$	R^*_a
R^*_a	$R^\backslash_a$	$R^/_a$	$L^\backslash_a$	$L^/_a$	L^*_a
$L^/_a$	$R^/_a$	L^*_a	R^*_a	$L^\backslash_a$	$R^\backslash_a$
$R^/_a$	$L^/_a$	R^*_a	L^*_a	$R^\backslash_a$	$L^\backslash_a$
$L^\backslash_a$	L^*_a	$R^\backslash_a$	$L^/_a$	R^*_a	$R^/_a$
$R^\backslash_a$	R^*_a	$L^\backslash_a$	$R^/_a$	L^*_a	$L^/_a$

Q	Q_1	Q_2	Q_3	Q_4	Q_5
(1)	(3)	(5)	(6)	(4)	(2)
(2)	(6)	(4)	(3)	(5)	(1)
(3)	(1)	(6)	(5)	(2)	(4)
(4)	(5)	(2)	(1)	(6)	(3)
(5)	(4)	(1)	(2)	(3)	(6)
(6)	(2)	(3)	(4)	(1)	(5)

Table 1. Images of Birkhoff's identities under parastrophies

Construction. We construct the algebra as a parastrophe of an algebra Z. Looking at Table 1, if our hypothetical algebra is a parastrophe, let us say Z_1, of the algebra Z, then Z has to satisfy (2), (3), (4) and (6). Such an algebra Z is a cancellative right quasigroup which is not left divisible [5, Example 3]. There are many such examples known, for instance $Z = (\mathbb{Z}, *, /, \backslash)$ with the operations defined by

$$a * b = a + 2b, \qquad a/b = a - 2b, \qquad a\backslash b = \left\lfloor \frac{b-a}{2} \right\rfloor.$$

Its parastrophe Z_1, explicitely written down in Example 2, then satisfies (1), (2), (5), (6) and not (3), (4). □

Problem 6. *Construct an algebra satisfying* (3), (4), (5), (6) *and not* (1), (2).

Construction. We construct the algebra as a parastrophe of an algebra Z. Looking at Table 1, if this hypothetical algebra is a parastrophe, let us say Z_1, of an algebra Z, then Z has to satisfy (1), (2), (4), (5) and not (3), (6). Such an algebra Z is a divisible left quasigroup which is not right cancellative [5, Example 4]. There are many such examples known, for instance $Z = (\mathbb{Z}, *, /, \backslash)$ with the operations defined by

$$a * b = a + \lfloor b/2 \rfloor, \qquad a/b = a - \lfloor b/2 \rfloor, \qquad a\backslash b = 2(b-a).$$

Its parastrophe Z_1, explicitely written down in Example 3, then satisfies (3), (4), (5), (6) and not (1), (2). □

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