ON A PARTIAL SYNTACTICAL CRITERION FOR THE LEFT DISTRIBUTIVITY AND THE IDEMPOTENCY

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ABSTRACT. We study here so called cuts of terms and their classes modulo the identities of the left distributivity and the idempotency. We give an inductive definition of such classes and this gives us a criterion that decides in some cases whether two terms are equivalent modulo both identities.

The article deals with the left distributive ($x \cdot yz = xy \cdot xz$) and the idempotent ($x = xx$) identities. The most natural example of a left distributive idempotent (LDI) groupoid is a group G with conjugation, *i.e.* the operation $x * y = x^{-1}yx$. However, it was proved independently by Larue [5] and Drápal, Kepka, Musílek [2] that there exist some terms that are not LDI-equivalent but they are equal if realised as the conjugation in a group.

So far, the only method, that enables us to distinguish two terms, not equivalent modulo LDI but equivalent modulo the group conjugation (GC), is the semantical criterion, *i.e.* finding a suitable *ad hoc* counterexample. This exactly was used in the two cited articles, they took a pair (both the same pair) of terms and a 4-element groupoid (both the same one) and mapped the terms at different elements of the groupoid.

The only groupoid example, that would give us a perfect answer for any pair of terms, saying if they are LDI-equivalent or not, is the free LDI-groupoid. However, its structure is unknown so far. Hence, to show that two terms are LDI-equivalent, we have to find its syntactical proof, and to show that two terms are not LDI-equivalent we either have to find a suitable counterexample or use a syntactical criterion, *i.e.* we have to find an invariant of the terms that is preserved modulo both identities.

So far, the only non-trivial invariant known is the weight of terms (see Section 3). In this article we present a criterion that gives different results than the weight, it means that it enables us to prove the non-equivalence of some pairs of terms with the same weight but on the other hand there are indistinguishable terms with different weights.

The article is organised as follows: in Section 1 we introduce basic definitions for the work with terms, like addresses and expansions. Section 2 is the core of the article. We present cuts of terms, techique invented by Dehornoy [1] and used to find a quasi-order on the set of terms, the equivalence of which is exactly the LD-equivalence (and thus the word problem of the free LD groupoid was solved). In our case, the quasi-order equivalence encompasses properly the equivalence generated by the left distributivity and the idempotency. Nevertheless, we can use it as a partial criterion to detect non-equivalence of terms. For this we describe inductively

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how any upper bounded set in this quasi-order looks like. This definition is not a construction in the sense that it does not enable us to enumerate all the terms belonging to the set. But if we find a common property of all elements of such a set, we can exclude effectively some terms from it. This is used in Section 3 to show an example of two terms that are not LDI-equivalent.

Most of the article was a part of the author's thesis [3].

1. Introductory definitions

In this section we introduce notations needed for the work with terms. The notations are standard and hence we do not explain them too carefully here, the reader can find thorough explanations in [1]. Recall that we work with binary terms.

Definition: An *address* is a finite sequence of 0 and 1. The empty address is denoted $\varnothing$. We say that an address α is a *prefix* of an address β, denoted $\alpha \sqsubseteq \beta$, if $\alpha = \beta\gamma$ for an address γ. We say that α is on the right of β, denoted $\alpha >_{\mathrm{LR}} \beta$, if $\gamma 1 \sqsubseteq \alpha$ and $\gamma 0 \sqsubseteq \beta$ for some address γ. We say that α is orthogonal to β, denoted $\alpha \perp \beta$, if α is on the right or on the left of β. We write $\alpha > \beta$ if $\alpha >_{\mathrm{LR}} \beta$ of $\alpha \sqsupseteq \beta$.

Note that $>$ is a linear order on the set of addresses.

Definition: Let t be a term and α an address. The *subterm* of t at α, or an α-subterm of t, is the term $\mathrm{sub}(t,\alpha)$ defined as

$$\mathrm{sub}(t,\alpha) = \begin{cases} t & \text{for } \alpha = \varnothing, \\ \mathrm{sub}(t_1,\beta) & \text{for } \alpha = 0\beta \text{ and } t = t_1 \cdot t_2, \\ \mathrm{sub}(t_2,\beta) & \text{for } \alpha = 1\beta \text{ and } t = t_1 \cdot t_2. \end{cases} \tag{1}$$

Definition: Let t be a term. We say that an address α lies in t if $\mathrm{sub}(t,\alpha)$ exists. In this case we say that α is an *external* address if $\mathrm{sub}(t,\alpha)$ is a variable and that α is *internal* otherwise. The *skeleton* of t is defined as the set $\mathrm{Skel}(t)$ of all addresses in t and the *outline* of t as the set $\mathrm{Out}(t)$ of all external addresses.

Notation. Let α be an address in t. Then there exist unique numbers p and q such that $\alpha 0^p$ and $\alpha 1^q$ are external. If there is no confusion we write $\alpha 0^*$ and $\alpha 1^*$ instead to avoid introducing p and q.

The aim of the article is to describe a syntactical criterion for the left distributivity and the idempotency. We denote $t \overset{\mathrm{LDI}}{=} t'$ the equivalence relation generated by the left distributive and the idempotent law, that means by $x \cdot yz \overset{\mathrm{LD}}{=} xy \cdot xz$ and $x \overset{\mathrm{I}}{=} xx$. We look at the identities as being a rewriting system:

Definition: We say that a term t' is a *basic LD-expansion* of a term t if t' is obtained from t by replacing a subterm of form $t_1 \cdot (t_2 \cdot t_3)$ by the term $(t_1 \cdot t_2) \cdot (t_1 \cdot t_3)$. We say that a term t' is a *basic I-expansion* of a term t if t' is obtained from t by replacing a subterm t_1 by the term $t_1 \cdot t_1$.
We say that a term t' is a *basic expansion* of a term t if t' is a basic LD-expansion or a basic I-expansion of the term t.
We say, for $k \geqslant 0$, that a term t' is a *k-expansion* of a term t (or simply an *expansion*, denoted $t \to t'$) if there exists a sequence $t = t_0, \ldots, t_k = t'$ of terms such that t_i is a basic expansion of t_{i-1}, for each $1 \leqslant i \leqslant k$.

The chosen rewriting system is evidently not finitely terminating, nevertheless we can prove its confluence [5].

Definition (iterated left subterm)**:** For two terms t_1 and t_2 we write $t_1 \sqsubset t_2$ if there exists $p > 0$ such that $t_1 = \mathrm{sub}(t_2, 0^p)$. Analogously, $t_1 \sqsubseteq t_2$ is the same relation with $p \geqslant 0$. We write $t_1 \sqsubseteq_{\mathrm{LDI}} t_2$ if there exist $t_1' \overset{\mathrm{LDI}}{=} t_1$ and $t_2' \overset{\mathrm{LDI}}{=} t_2$ with $t_1' \sqsubseteq t_2'$.

The $\sqsubseteq$ relation is a partial order. The $\sqsubseteq_{\mathrm{LDI}}$ relation is a quasi-order; it is easy to prove that it is not an order: we have $xy \sqsubset xy \cdot x \sqsubset (xy \cdot x) \cdot (xy \cdot y)$. Since $xy \to (xy \cdot x) \cdot (xy \cdot y)$, we have $xy \sqsubseteq_{\mathrm{LDI}} xy \cdot x \sqsubseteq_{\mathrm{LDI}} xy$. However, these two terms cannot be equivalent since both LD and I identities preserve the rightmost variable.

The iterated left subterms are, modulo LDI-equivalence, preserved by expansions:

Proposition 1.1: *Let t' be a k-expansion of a term t. If the address 0^p is in t, for some $p \geqslant 0$, then there exist $p' \geqslant p$ and $k' \leqslant k$ such that $\mathrm{sub}(t', 0^{p'})$ is a k'-expansion of* $\mathrm{sub}(t, 0^p)$.

Proof. We can suppose that t' is a basic expansion of t. We show the result by induction on t. For t a variable, the result is true. Suppose $t = t_1 \cdot t_2$ and $t' = t_1' \cdot t_2'$. If $p = 0$ then the result is trivial. Hence suppose $p > 0$. We have three possibilities: if the basic expansion was $t_2 \to t_2'$ then $\mathrm{sub}(t, 0^p) = \mathrm{sub}(t', 0^p)$. If the basic expansion was $t_1 \to t_1'$ then we use the induction hypothesis. Finally suppose that the expansion was made in the root. Denote $t_2 = t_3 \cdot t_4$. Then we have either $t' = (t_1 \cdot t_3) \cdot (t_1 \cdot t_4)$ or $t' = (t_1 \cdot t_2) \cdot (t_1 \cdot t_2)$ and the result is clear. □

2. Cuts of terms

In this section we introduce the main tool of the article—the cuts of terms. We study the connection between cuts of terms and left iterated subterms and this study leads to an inductive description of all the terms s with $s \sqsubseteq_{\mathrm{LDI}} t$, for a given term t.

Definition: Let α be an address in a term t. We define the *cut of t in α* as the term $\mathrm{cut}(t, \alpha)$ recursively:

$$\mathrm{cut}(t,\alpha) = \begin{cases} t & \text{for } \alpha = \varnothing, \\ \mathrm{cut}(t_1, \beta) & \text{for } \alpha = 0\beta \text{ and } t = t_1 \cdot t_2, \\ t_1 \cdot \mathrm{cut}(t_2, \beta) & \text{for } \alpha = 1\beta \text{ and } t = t_1 \cdot t_2. \end{cases} \tag{2}$$

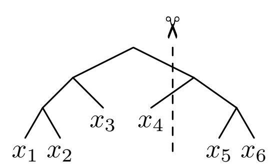

Figure 1. The cut of the term $((x_1 \cdot x_2) \cdot x_3) \cdot (x_4 \cdot (x_5 \cdot x_6))$ in the address 10 is the term $((x_1 \cdot x_2) \cdot x_3) \cdot x_4$.

Example: To make a cut if term t in an external address α means to cut the tree of t right after the leaf with the address α. We remove the right part and reconstruct the term with the remainder on the left (see Figure 1). Consider $t = ((x_1 \cdot x_2) \cdot x_3) \cdot (x_4 \cdot (x_5 \cdot x_6))$. One has $\mathrm{cut}(t, 000) = x_1$, $\mathrm{cut}(t, 001) = x_1 \cdot x_2$, $\mathrm{cut}(t, 01) = (x_1 \cdot x_2) \cdot x_3$, $\mathrm{cut}(t, 10) = ((x_1 \cdot x_2) \cdot x_3) \cdot x_4$, $\mathrm{cut}(t, 110) = ((x_1 \cdot x_2) \cdot x_3) \cdot (x_4 \cdot x_5)$ and $\mathrm{cut}(t, 111) = t$.

The example explains how to understand cuts in the external addresses. For the internal ones, we have

Lemma 2.1 (in [1])**:** *Let t be a term. Then, for each $\alpha \in \mathrm{Skel}(t)$, we have* $\mathrm{cut}(t,\alpha) = \mathrm{cut}(t,\alpha 1^*)$.

We can see easily that $s \sqsubseteq t$ means that s is a cut of t. The other direction is also true, up to LDI-equivalence:

Lemma 2.2 (in [1])**:** *Let s be a cut of a term t at an address α. Then there exists t', an LD-expansion of t, with $s \sqsubseteq t'$.*

What happens with cuts if we make different expansions? Some new cuts can appear while all cuts of the starting term are expanded to cuts of the expanded term:

Lemma 2.3: *Let t' be an expansion of a term t. Let α be an address in t. Then there exists α', an address in t' such that* $\mathrm{cut}(t,\alpha) \to \mathrm{cut}(t',\alpha')$.

Proof. We can suppose that t' is a basic expansion of t at an address β. We can also suppose that α is an external address. If $\alpha \perp \beta$ then evidently $\mathrm{cut}(t,\alpha) \to \mathrm{cut}(t',\alpha)$. Hence suppose $\beta \sqsubseteq \alpha$. For an LD-expansion in β, we have $\mathrm{cut}(t,\beta 0\gamma) = \mathrm{cut}(t',\beta 00\gamma)$, $\mathrm{cut}(t,\beta 10\gamma) = \mathrm{cut}(t',\beta 01\gamma)$ and $\mathrm{cut}(t,\beta 11\gamma) \to \mathrm{cut}(t',\beta 11\gamma)$, for all addresses γ. For an I-expansion in β, we have $\mathrm{cut}(t,\beta\gamma) = \mathrm{cut}(t',\beta 0\gamma)$, for each address γ. □

There is a question: how can we describe those new cuts of the expanded term? To answer this, we need an auxiliary observation. The expression $x \cdot y \cdot z$ means $x \cdot (y \cdot z)$.

Lemma 2.4 (in [1])**:** *Let α be an address in a term t. Then*

$$\mathrm{cut}(t,\alpha) = \mathrm{sub}(t,\alpha_1 0) \cdot \mathrm{sub}(t,\alpha_2 0) \cdot \cdots \cdot \mathrm{sub}(t,\alpha_p 0) \cdot \mathrm{sub}(t,\alpha), \tag{3}$$

where $\alpha_1, \ldots, \alpha_p$ are the $\sqsubset$-ordered set of all the prefixes of α such that $\alpha_1 1, \ldots, \alpha_p 1$ are prefixes of α too.

Proposition 2.5: *Let t' be a basic expansion of a term t and let α' be an address in t. Then one of the following possibilities holds:*
- there exists α, an address in t, such that $\mathrm{cut}(t,\alpha) \to \mathrm{cut}(t',\alpha')$*;*
- there exist addresses $\alpha_1 \geqslant \alpha_2$ in $\mathrm{Out}(t)$, such that $\mathrm{cut}(t',\alpha') \to \mathrm{cut}(t,\alpha_1) \cdot \mathrm{cut}(t,\alpha_2)$.

Proof. We can suppose α external. Let t' be a basic expansion of t at an address β. According to the proof of Lemma 2.3, all configurations but two fall in the first possibility. Let us investigate the remaining two.

If the expansion is the basic LD-expansion at β and $\alpha' = \beta 10\gamma$, for an address γ, then, according to Lemma 2.4,

$$\begin{aligned}
\mathrm{cut}(t',\alpha') &= \mathrm{sub}(t',\beta_1 0)\cdots \mathrm{sub}(t',\beta_p 0)\cdot \mathrm{sub}(t',\beta 0)\\
&\qquad \cdot \mathrm{sub}(t',\beta 10\gamma_1 0)\cdots \mathrm{sub}(t',\beta 10\gamma_q 0)\cdot \mathrm{sub}(t',\beta 10\gamma)\\
&= \mathrm{sub}(t,\beta_1 0)\cdots \mathrm{sub}(t,\beta_p 0)\cdot (\mathrm{sub}(t,\beta 0)\cdot \mathrm{sub}(t,\beta 10))\\
&\qquad \cdot \mathrm{sub}(t,\beta 0\gamma_1 0)\cdots \mathrm{sub}(t,\beta 0\gamma_q 0)\cdot \mathrm{sub}(t,\beta 0\gamma)\\
&\to \mathrm{sub}(t,\beta_1 0)\cdots \big(\mathrm{sub}(t,\beta_p 0)\cdot (\mathrm{sub}(t,\beta 0)\cdot \mathrm{sub}(t,\beta 10))\big)\cdot \mathrm{sub}(t,\beta_p 0)\\
&\qquad \cdot \mathrm{sub}(t,\beta 0\gamma_1 0)\cdots \mathrm{sub}(t,\beta 0\gamma_q 0)\cdot \mathrm{sub}(t,\beta 0\gamma)\\
&\to \big(\mathrm{sub}(t,\beta_1 0)\cdots \mathrm{sub}(t,\beta_p 0)\cdot \mathrm{sub}(t,\beta 0)\cdot \mathrm{sub}(t,\beta 10)\big)\cdot \big(\mathrm{sub}(t,\beta_1 0)\\
&\qquad \cdots \mathrm{sub}(t,\beta_p 0)\cdot \mathrm{sub}(t,\beta 0\gamma_1 0)\cdots \mathrm{sub}(t,\beta 0\gamma_q 0)\cdot \mathrm{sub}(t,\beta 0\gamma)\big)\\
&= \mathrm{cut}(t,\beta 10)\cdot \mathrm{cut}(t,\beta 0\gamma) = \mathrm{cut}(t,\beta 101^*)\cdot \mathrm{cut}(t,\beta 0\gamma),
\end{aligned}$$

where $\beta_1,\ldots,\beta_p$ and $\gamma_1,\ldots,\gamma_q$ have the same meaning as in Lemma 2.4.

If the expansion is the basic I-expansion at β and $\alpha' = \beta 1\gamma$, for some address γ, then a similar reasoning gives $\mathrm{cut}(t',\alpha') \to \mathrm{cut}(t,\beta 1^*)\cdot \mathrm{cut}(t,\beta\gamma)$. □

The proposition gives raise to the definition of the set of all such terms that can appear as cuts of terms equivalent to a term t:

Definition: For a term t, we define $\mathrm{Cut}(t)$ as the smallest set of terms satisfying:
1) each cut of t belongs to t;
2) if a term s' is equivalent to a term s from $\mathrm{Cut}(t)$ then s' belongs to $\mathrm{Cut}(t)$ too;
3) let s and s' belong to $\mathrm{Cut}(t)$, if there exists a term t', equivalent to t, whom s is the cut at an external address α and s' is the cut at an external address α' and if $\alpha \geqslant \alpha'$ then the term $s \cdot s'$ belong to $\mathrm{Cut}(t)$.

Corollary 2.6: *Let t and t' be two equivalent terms. Then each cut of t' belongs to* $\mathrm{Cut}(t)$.

Proof. Use induction on the length of the proof $t \overset{\text{LDI}}{=} t'$, together with Lemma 2.3 and Proposition 2.5. □

The set $\mathrm{Cut}(t)$ is supposed to be the set of all the cuts of all the terms equivalent to t. We have proved one direction only, the other direction comes immediately:

Proposition 2.7: *Let t be a term. For each $s \in \mathrm{Cut}(t)$ there exists a term t', equivalent to t, such that s is a cut of t'.*

Proof. The set $\mathrm{Cut}(t)$ is build up inductively. For any term added to $\mathrm{Cut}(t)$ by the first or by the second rule, the proposition is trivial. Hence suppose that we have a term s added by the third rule, *i.e.,* one has $s = s_1 \cdot s_2$ with $s_1, s_2 \in \mathrm{Cut}(t)$ and both subterms satisfy the induction hypothesis. Then there exists a term t_1, equivalent to t, with s_1 and s_2 as cuts. Moreover, the address of the cut s_1 is on the right of the address of the cut s_2 and therefore s_2 can be seen as a cut of s_1.

According to Lemma 2.2, “being a cut of” is a subrelation of the relation $\sqsubseteq_{\text{LDI}}$. We can thus rewrite the situation as $s_2 \sqsubseteq_{\text{LDI}} s_1 \sqsubseteq_{\text{LDI}} t_1$. By definition, there exist $s_1' \overset{\text{LDI}}{=} s_1$, $s_2' \overset{\text{LDI}}{=} s_2$ and $t_2 \overset{\text{LDI}}{=} t_1$ such that $s_2' \sqsubseteq s_1' \sqsubseteq t_2$. We denote by α the address of s_1' in t_2 and by $\alpha\beta$ the address of s_2' in t_2. We denote by t_3 the term obtained from t_2 by making the basic I-expansion at α. We have then

$$\mathrm{cut}(t_3,\alpha 1\beta) = \mathrm{sub}(t_3,\alpha)\cdot \mathrm{sub}(t_3,\alpha 1\beta) = \mathrm{sub}(t_2,\alpha)\cdot \mathrm{sub}(t_2,\alpha\beta) = s_1'\cdot s_2' \overset{\text{LDI}}{=} s.$$

Since t_3 is equivalent to t, we have $s \sqsubseteq_{\text{LDI}} t$. Replacing the term $s_1' \cdot s_2'$ in t_3 by s, we obtain an equivalent term t' with $s \sqsubseteq t'$, making a contradiction. □

We can gather all the information gained in this section into the following theorem:

Theorem 2.8: *The following conditions are equivalent for two terms s and t:*
(i) $s \sqsubseteq_{\text{LDI}} t$;
(ii) there exists a term t' with $t' \overset{\text{LDI}}{=} t$ and $s \sqsubseteq t$;
(iii) there exists a term t' with $t' \overset{\text{LDI}}{=} t$ and $\alpha \in \operatorname{Skel}(t)$ such that $s = \operatorname{cut}(t', \alpha)$;
(iv) $s \in \operatorname{Cut}(t)$;
(v) $\operatorname{Cut}(s) \subseteq \operatorname{Cut}(t)$.

Proof. (i)⇒(ii) The definition says that there exist $s' \overset{\text{LDI}}{=} s$ and $t'' \overset{\text{LDI}}{=} t'$ such that $s' \sqsubseteq t''$. The term t' is obtained from t'' replacing the subterm s' by s. (ii)⇒(iii) Evident. (iii)⇒(i) Lemma 2.2. (iii)⇒(iv) Corollary 2.6. (iv)⇒(iii) Proposition 2.7. (iv)+(i)⇒(v) Due transitivity of $\sqsubseteq_{\text{LDI}}$. (v)⇒(iv) Evident. □

Remark. We can now write the third rule in the definition of $\operatorname{Cut}(t)$ more briefly: if $s' \sqsubseteq_{\text{LDI}} s$ both belong to $\operatorname{Cut}(t)$ then $s \cdot s'$ belongs to $\operatorname{Cut}(t)$ too.

3. The criterion

In this section we describe a syntactical method that enables us to distinguish, in some cases, two non-equivalent terms. This method uses the weight of terms:

Definition: Let us choose a real number $p \in [0, 1]$ and real numbers w_x, for each variable x. Then the *weight* of a term t is defined inductively:

$$w(t) = \begin{cases} w_x & \text{for } t = x, \text{ a variable,} \\ p \cdot w(t_1) + (1-p) \cdot w(t_2) & \text{for } t = t_1 \cdot t_2. \end{cases} \tag{4}$$

It is easy to show that two equivalent terms have the same weight, whatever constants we choose. It is also easy to find two non-equivalent terms with the same weight but, of course, one has to find a different criterion how to prove the non-equivalence of the terms. The criterion we will discuss here is

$$\text{If } t \overset{\text{LDI}}{=} t' \text{ then } \big(t \sqsubseteq_{\text{LDI}} t' \text{ and } t' \sqsubseteq_{\text{LDI}} t\big). \tag{5}$$

We have already shown in Section 1 that the condition in (5) is necessary but not sufficient. Moreover, the example terms have different rightmost variables and hence their non-equivalence is detected by nearly any weight. Nevertheless, there exist couples of terms distinguishable by our criterion and indistinguishable by weights. An example we show here is xy and $(x \cdot xy) \cdot (yx \cdot y)$.

Lemma 3.1: *For any choice of constants, the weight of xy is the same as the weight of $(x \cdot xy) \cdot (yx \cdot y)$.*

Proof. $w((x \cdot xy) \cdot (yx \cdot y)) =$

$$= p^2 w_x + p^2(1-p)w_x + p(1-p)^2 w_y + p^2(1-p)w_y + p(1-p)^2 w_x + (1-p)^2 w_y$$
$$= p^2 w_x + p(1-p)w_x + p(1-p)w_y + (1-p)^2 w_y = pw_x + (1-p)w_y = w(xy). \quad \square$$

These two terms are indistinguishable using the weight criterion. We want to distinguish them using our criterion, more precisely, we want to show $(x \cdot xy) \cdot (yx \cdot y) \not\sqsubseteq_{\text{LDI}} xy$. Using Theorem 2.8 (v) we want to show that there exists a term in $\mathrm{Cut}((x \cdot xy) \cdot (yx \cdot y))$ not belonging to $\mathrm{Cut}(xy)$. For this we find a common property of all terms in $\mathrm{Cut}(xy)$—a weight.

Lemma 3.2: *Let us set $w_x = 1$, $w_y = -1$ and $p = \frac{1}{2}$. Then each term from $\mathrm{Cut}(xy)$ has non-negative weight.*

Proof. We show the result for all three rules of $\mathrm{Cut}(xy)$ construction.
1) Weights of cuts: $w(x) = 1$, $w(xy) = 0$.
2) Equivalent terms have the same weights.
3) If $w(s) \geqslant 0$ and $w(s') \geqslant 0$ then $w(s \cdot s') \geqslant 0$. □

It remains to find a term t from $\mathrm{Cut}((x \cdot xy) \cdot (yx \cdot y))$ with $w(t) < 0$.

Proposition 3.3: *The terms xy and $(x \cdot xy) \cdot (yx \cdot y)$ are not equivalent.*

Proof. Let us take $t = (x \cdot xy) \cdot y$, which is $\mathrm{cut}((x \cdot xy) \cdot (yx \cdot y), 100)$. Then $w(t) = -\frac{1}{4}$ and, according to Lemma 3.2, the term t cannot belong to $\mathrm{Cut}(xy)$. Therefore, according to Theorem 2.8, $(x \cdot xy) \cdot (yx \cdot y) \not\sqsubseteq_{\text{LDI}} xy$ and these terms are not equivalent. □

4. Open problem

In the introduction, we have spoken about the *group conjugation*, a proper subvariety of LDI variety. The free algebra of the group conjugation variety is a subgroupoid of the free group with conjugation [6] and therefore the word problem in this algebra is easy to solve. Hence the difficulty of the word problem of the free LDI groupoid lies outside of the group conjugation. Thus one natural question arises: is the criterion of this article helpful when dealing with GC-equivalent terms?

The shortest known example of GC-equivalent terms that are not LDI-equivalent is $(xy \cdot y)x$ and $xy \cdot (yx \cdot x)$ [5] and [2]. A straitforward calculation shows $\mathrm{Cut}((xy \cdot y)x) = \mathrm{Cut}(xy \cdot (yx \cdot x))$; the calculation was done in [3]. Nevertheless, it seems that the variety generated by the group conjugation is not finitely based and hence there might be enough space to find a different pair of GC-equivalent terms where our criterion applies to show their LDI-non-equivalence.

Open Problem 4.1: For two terms t and t', does $t \overset{\text{GC}}{=} t'$ imply $t \sqsubseteq_{\text{LDI}} t'$ or not?

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