A COMBINATORIAL CONSTRUCTION OF THE WEAK ORDER OF A COXETER GROUP

Přemysl Jedlička
Université de Caen,
Laboratoire LMNO, Caen,
France

Abstract

Let W be a (finite or infinite) Coxeter group and W_X be a proper standard parabolic subgroup of W. We show that the semilattice made up by W equipped with the weak order is a semidirect product of two smaller semilattices associated with W_X.

1 INTRODUCTION

Let W be a Coxeter group. The weak order on W, considered in [2], is a partial order $\preccurlyeq$ on W which forms a meet-semilattice, *i.e.*, each pair of elements of W admits a greatest lower bound. When its edges are labelled by the generators, the Hasse diagram of this semilattice is the Cayley graph of W with respect to the standard presentation.

For each standard parabolic subgroup W_X of W, there exists a natural decomposition of the elements of W involving W_X. The point of this paper is to show that this decomposition preserves the weak order well enough to reconstruct the weak order of W from that of W_X together with certain data on the group W_X.

We use a general construction called *semidirect product* of semilattices, which was introduced in [7] and investigated in [8], to determine the semilattice of W starting from two smaller semilattices. This result is as follows:

Theorem: *Let (W, S) be a Coxeter system and let X be a subset of S. Let W_X be the standard parabolic subgroup of W associated with X and let W^X be the set of shortest-leght right coset representatives relative to W_X. Then the meet-semilattice $(W, \preccurlyeq)$ is isomorphic to the semidirect product $(W_X, \preccurlyeq) \ltimes_\psi (W^X, \preccurlyeq)$, where ψ is some explicit mapping from W_X^2 to $\mathrm{End}(W^X)$.*

As an application, we deduce an algorithm for computing the lattice operations, *i.e.*, the infimum and the supremum of two elements.

In the case of a finite Coxeter group, a similar inductive construction was developed by Le Conte de Poly-Barbut in [10] from a different approach. She shows that the weak order of W can be constructed starting from the subgroup W_X, the set W^X and a certain mapping from $W^X \times X$ to $S \cup \{\emptyset\}$. One advantage of our approach is that it enables us to cover the infinite case as well, at the expense of using the notion of a semidirect product of semilattices. Another advantage is that our approach does not only give an abstract description of the weak order but it also gives an algorithmic method for computing the (semi)lattice operations.

Another decomposition appears in [5] too, where the Cayley graph of W is expressed as a semidirect product of two smaller graphs, corresponding to W_X and W^X as above. However, contrary to the approach of [10] and the current approach, this approach only describes the relation between the larger graph and the smaller ones, and does not enable one to reconstruct the larger one from its parts.

The paper is organized as follows: in Section 2 we recall some basic facts about the weak order and standard parabolic subgroups in a Coxeter group. In Section 3 we give a brief general description of the semidirect product of (semi)lattices. This definition is used in Section 4 to present our construction of the Cayley graphs of Coxeter groups. Finally, in Section 5 we develop some examples of our construction, namely the types B_n and $\tilde{A}_2$.

2 THE WEAK ORDER OF COXETER GROUPS

Definition 2.1: Let W be a group generated by a finite set S. For all s, t in S, let us denote by $m_{s,t}$ the order of the element st in W; we write $m_{s,t} = \infty$ if st is of infinite order. We say that W is a *Coxeter group* if it admits a presentation of the form

$$\langle\, S;\ s^2 = 1,\ (st)^{m_{s,t}} = 1,\ \text{for all } s, t \text{ in } S \text{ and } m_{s,t} < \infty \,\rangle.$$

For each element g in W, we define the *length* $\ell(g)$ of g to be the least k such that g can be written as $s_1 s_2 \ldots s_k$, with all s_i in S. We then call such a decomposition a *reduced decomposition* of g. This length has some nice properties: We have $\ell(g) = 0$ if and only if g is the neutral element. We have also $\ell(g) = \ell(g^{-1})$, and for each g, h in W we have the inequality $\ell(g) + \ell(h) \geqslant \ell(gh)$.

Definition 2.2: Let W be a Coxeter group. For g, h in W, we write $g \preccurlyeq h$ if we have $\ell(g) + \ell(g^{-1}h) = \ell(h)$. This relation is called the *weak order*.

We can equivalently define this order saying: $g \preccurlyeq h$ holds if there exist u, a reduced decomposition of g, and $s_1, s_2, \ldots, s_k$ in S such that $us_1 s_2 \ldots s_k$ is a reduced decomposition of h. Regarding this definition it is obvious that the relation $\preccurlyeq$ is a partial order. In the following text the words: "g is less than h", for g, h two elements of a Coxeter group, will always refer to the weak order so defined.

Proposition 2.3 [2]: *Let W be a Coxeter group. Then the set $(W, \preccurlyeq)$ is a meet-semilattice. The element 1 is its least element. Moreover, if W is finite, then $(W, \preccurlyeq)$ is a lattice.*

We denote by $\vee$ and $\wedge$ the (partial) lattice operations associated with the weak order, *i.e.*, we define $g \wedge h$ to be $\max\{\, a \in W;\ a \preccurlyeq g \text{ and } a \preccurlyeq h \,\}$, and $g \vee h$ to be $\min\{\, a \in W;\ g \preccurlyeq a \text{ and } h \preccurlyeq a \,\}$, if the set is nonempty.

Notation 2.4: We write $[s, s'\rangle^m$ for $\underbrace{ss'ss'\cdots}_{m \text{ factors}}$ and $\langle s, s']^m$ for $\underbrace{\cdots ss'ss'}_{m \text{ factors}}$.

Lemma 2.5 [3]: *Let g be in W, and let $s \neq s'$ be in S.*
(i) If $\ell(gs) = \ell(gs') = \ell(g) + 1$ holds, then $(gs) \vee (gs')$ exists if and only if $m_{s,s'}$ is finite; in this case we have $(gs) \vee (gs') = g[s, s'\rangle^{m_{s,s'}}$.
(ii) If $\ell(gs) = \ell(gs') = \ell(g) - 1$ holds, then $m_{s,s'}$ is finite and we have $(gs) \wedge (gs') = g[s, s'\rangle^{m_{s,s'}}$.

We denote by $\prec$ the strict weak order, *i.e.*, $a \prec b$ is equivalent to $a \preccurlyeq b$ and $a \neq b$. We say that an element b *covers* an element a in W if b is an immediate successor of a, *i.e.*, we have $a \prec b$ and there is no other element c in W verifying $a \prec c \prec b$. We observe that, in the set $(W, \preccurlyeq)$, an element h covers g only if $h = gs$ holds, for some s in S. Conversely, if we have $h = gs$, for some s in S, then either h covers g or g covers h. Hence, the Hasse diagram of $(W, \preccurlyeq)$ is nothing but the unlabelled Cayley graph of W. So, when we construct a Hasse diagram of the lattice $(W, \preccurlyeq)$ and we label each edge by the appropriate generator, we obtain the Cayley graph of the group W.

When we say "the Cayley graph of a Coxeter group W", we mean the Cayley graph of W with respect to the presentation (2.1). Because all generators in this presentation are involutive, for each edge there exists a backward edge labelled by the same generator. That is why it is common to draw unoriented edges (see Figure 1).

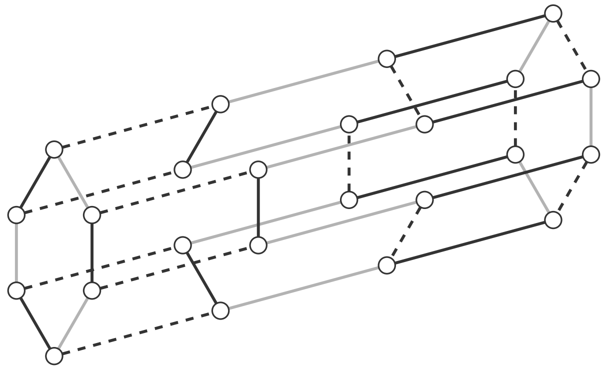

Figure 1: Example—the Cayley graph of $\mathfrak{S}_4$ drawn as the Hasse diagram of the weak order: The black edges represent the transposition (1,2), the gray ones represent the transposition (2,3) and the dashed ones represent the transposition (3,4).

Now we recall some basic facts about standard parabolic subgroups, namely a decomposition associated with a parabolic subgroup.

Proposition 2.6 [6]: *Let $W = \langle S; R\rangle$ be a Coxeter group. Let X be a subset of S. Then the subgroup generated by X is a Coxeter group generated by X as the set of Coxeter generators.*

The subgroup mentioned in Proposition 2.6 is usually denoted W_X and called a *standard parabolic subgroup* of W generated by X.

It is useful to mention that the definition of ℓ does not depend on the group, in the sense that for each g in W_X the lengths in W and in W_X are the same and we need not distinguish them. The same is true for the weak order.

In the sequel, we fix a Coxeter group W with the Coxeter presentation $\langle S; R\rangle$, we fix a subset X of S and we let W_X be the associated standard parabolic subgroup of W.

Definition 2.7: We say that an element g in W is *X-reduced* if $\ell(xg) = \ell(g) + 1$ holds for each x in X. We denote by W^X the set of all X-reduced words in W.

The following result is standard (see [3]):

Lemma 2.8: *Let g be an element of W. Then the following conditions are equivalent:*
(i) The element g is X-reduced;
(ii) The element g has the shortest length among the class $W_X g$;
(iii) The element g is the only one with the shortest length among the class $W_X g$;
(iv) For each h in W_X we have $\ell(hg) = \ell(h) + \ell(g)$;
(v) For each h in W, whenever $h \preccurlyeq g$ holds then h is X-reduced;

For each element g in W we can find exactly one element g'' which is the only X-reduced element of the set $W_X g$. If we denote gg''^{-1} by g', we have a unique decomposition of g of the form $g = g'g''$, where g' belongs to W_X and g'' belongs to W^X.

Definition 2.9: We define mappings $\alpha_X : W \to W_X$ and $\omega_X : W \to W^X$ so that $g = \alpha_X(g)\omega_X(g)$ holds for each g in W.

Using Definition 2.9 and Lemma 2.8 we obtain the following proposition:

Proposition 2.10: *The mapping $g \mapsto \big(\alpha_X(g), \omega_X(g)\big)$ is a bijection between the set W and the set $W_X \times W^X$.*

3 SEMIDIRECT PRODUCTS OF SEMILATTICES

In this section we describe a general (semi)lattice construction called *semidirect product.* It is an analog of the semidirect product of the group theory. We shall not give here a detailed proof of all properties as this is not needed for our current purposes. A more complete study can be found in [8]. We recall that the set $\mathrm{End}(H)$, for a meet-semilattice H, is the set of all its meet-semilattice endomorphisms, *i.e.*, the set of all mappings from H to H compatible with the meet operation.

Proposition 3.1 [8]: *Let K, H be two meet-semilattices and let ψ be a mapping from $K \times K$ to $\mathrm{End}(H)$ satisfying for all k, k', k'' in K:*

$$\psi_{k,k} = \mathrm{id}_H \tag{1}$$

$$\psi_{k,k' \wedge k''} = \psi_{k \wedge k', k''} \circ \psi_{k,k'} \tag{2}$$

Then the set $K \times H$ equipped with the operation $\wedge$ defined as

$$(k, h) \wedge (k', h') = (k \wedge k', \psi_{k,k'}(h) \wedge \psi_{k',k}(h')), \tag{3}$$

is a meet-semilattice. This semilattice is denoted by $K \ltimes_\psi H$.

It is well-known that the semidirect product of groups can be introduced both from an internal and an external point of view, according to whether we wish to decompose a group in terms of subgroups, or to construct a new group starting from two groups. The same is true for the semidirect product of semilattices. If we start with the semidirect product $K \ltimes_\psi H$, we can recover K and H as follows: consider the relation $\sim$ such that $(k, h) \sim (k', h')$ means $k = k'$; this relation is a congruence, its quotient is isomorphic to K and all congruence classes are isomorphic to H. But we can also start from the other end, *i.e.*, we can start with a lattice equipped with a specific congruence.

Proposition 3.2 [8]: *Let L be a meet-semilattice and let θ be a congruence on L. Let K denote the factor lattice $L/_\theta$. Assume that each class of congruence is isomorphic to a meet-semilattice H and that, for each h in H, the set $\{h' \in H;\ (\exists\, k, k' \in K)\,((k, h) \geqslant (k', h'))\}$ has an upper bound. Then there exists a mapping ψ from $K \times K$ to $\mathrm{End}(H)$, satisfying the conditions (1) and (2), such that the semidirect product $K \ltimes_\psi H$ is isomorphic to the lattice L.*

Here we intend to work with the weak order semilattices in Coxeter groups, which have a good property, namely that each interval is finite. In this case we have the following additional results:

Lemma 3.3 [8]: *Let K, H be meet-semilattices and let ψ be a mapping from $K \times K$ to $\mathrm{End}(H)$ such that $K \ltimes_\psi H$ exists.*
(i) For each $k \leqslant k'$ in K, the mapping $\psi_{k,k'}$ acts identically.
(ii) For each k' incomparable to k in K, we have $\psi_{k',k} = \psi_{k',k \wedge k'}$.
(iii) For each $k \leqslant k' \leqslant k''$ in K, we have $\psi_{k'',k} = \psi_{k',k} \circ \psi_{k'',k'}$.
(iv) If each interval in K is of finite length then the mapping ψ is uniquely defined by the mappings $\psi_{k',k}$, where k' covers k in K.
(v) If each interval in H is of finite length then, for all k, k', where k' covers k in K and each h in H, we have

$$\psi_{k',k}(h) = \psi_{k',k}\big(\max\{h' \in H :\ (h' \leqslant h) \text{ and } ((k', h') \text{ covers } (k, \psi_{k',k}(h')))\}\big).$$

How to understand the previous lemma? Suppose that we have a meet-semilattice $K \ltimes_\psi H$ with all its intervals finite. Then the mapping ψ is uniquely defined once we know when $(k, \psi_{k',k}(h'))$ is covered by (k', h') for all possible h' in H and all pairs k' covering k in K. If we have this little information on ψ then we can recover entire $\psi_{k',k}$ for an arbitrary pair k' covering k (Part (v)) and hence also all $\psi_{k',k}$ for all $k' \geqslant k$ (Parts (iii) and (iv)) and therefore arbitrary $\psi_{k',k}$ (Parts (i) and (ii)). Using different words, we can say that the mapping ψ is uniquely defined given a certain set of certain pairs $(k, h), (k', h')$. Actually, we are going to construct the Cayley graphs of Coxeter groups and therefore we want to know not only what covers what, *i.e.*, where are the edges in the Cayley graph, but also what generator we have to multiply by, to obtain the first element from the latter one, *i.e.*, we want also to know the label of the considered edge in the Cayley graph. Hence we consider certain triples $((k, h), (k', h'), s)$:

Definition 3.4: Let W be a Coxeter group with presentation $\langle S; R\rangle$ and let K, H be two sublattices of $(W, \preccurlyeq)$. We call a triple $((k, h), (k', h'), s)$ a *covering triple*, if k' covers k in K, the elements h, h' are in H and the element s from S satisfies $khs = k'h'$.

We will see later that we have $h = h'$ whenever $((k, h), (k', h'), s)$ is a covering triple.

In the sequel, we shall use a lattice version of the semidirect product. Here, two auxiliary mappings, $\varphi : K \times K \to \mathrm{End}((H, \vee))$ and $\psi : K \times K \to \mathrm{End}((H, \wedge))$, are necessary. We define the lattice operations $\vee$, $\wedge$ as follows:

$$\begin{aligned}(k, h) \vee (k', h') &= (k \vee k', \varphi_{k,k'}(h) \vee \varphi_{k',k}(h')),\\ (k, h) \wedge (k', h') &= (k \wedge k', \psi_{k,k'}(h) \wedge \psi_{k',k}(h')).\end{aligned}$$

Again, see [8] for the complete description. The semidirect product of the lattices K, H associated with φ, ψ is denoted $K \ltimes_\psi^\varphi H$.

Proposition 3.5 [8]: *Let L be a finite lattice and let $\sim$ be a congruence on L such that all its congruence classes are isomorphic. Let us denote by K the factor lattice $L/_\sim$ and by H one of the congruence classes. Then there exist mappings φ, ψ such that L is isomorphic to $K \ltimes_\psi^\varphi H$.*

4 SEMIDIRECT PRODUCTS IN COXETER GROUPS

Now we will investigate the connection between the mappings α_X and ω_X from Definition 2.9 and the weak order and show that these mappings are connected with a semidirect product of (semi)lattices.

The following lemma is a classical stuff:

Lemma 4.1 [4]: *Let h be in W^X and let s be in S. Then exactly one of the following three possibilities occurs:*

(i) $\ell(h) \geqslant \ell(hs)$*, in this case hs is X-reduced as well;*
(ii) $\ell(h) \leqslant \ell(hs)$ *and hs belongs to* W^X*;*
(iii) $\ell(h) \leqslant \ell(hs)$ *and hs not in* W^X*, in this case we have* $hs = s'h$ *for some* $s' \in X$*.*

Lemma 4.2: *Let* g, h *be in* W*. Then*
(i) $g \preccurlyeq h$ *implies* $\alpha_X(g) \preccurlyeq \alpha_X(h)$ *and* $\omega_X(g) \preccurlyeq \omega_X(h)$*;*
(ii) if h *covers* g *in* W *then we have either* $\alpha_X(g) = \alpha_X(h)$ *or* $\omega_X(g) = \omega_X(h)$*;*
(iii) we have $\alpha_X(g \wedge h) = \alpha_X(g) \wedge \alpha_X(h)$*.*

PROOF: Parts *(i)* and *(ii)* are consequences of Lemma 4.1.

(iii) According to Part *(i)* we have $\alpha_X(g \wedge h) \preccurlyeq \alpha_X(g) \wedge \alpha_X(h)$. However we have $\alpha_X(g) \preccurlyeq g$ and $\alpha_X(h) \preccurlyeq h$ and therefore $\alpha_X(g) \wedge \alpha_X(h) \preccurlyeq g \wedge h$ holds. Using α on this inequality we obtain $\alpha_X(\alpha_X(g) \wedge \alpha_X(h)) \preccurlyeq \alpha_X(g \wedge h)$. But α_X acts identically on W_X and therefore $\alpha_X(g) \wedge \alpha_X(h) \preccurlyeq \alpha_X(g \wedge h)$ holds. ■

Thus α_X is compatible with the meet, *i.e.*, it is an endomorphism of the semilattice $(W, \wedge)$. This mapping is, in the case of finite Coxeter group, compatible with the join too, *i.e.*, it is a lattice endomorphism as will be shown below (see Lemma 4.5).

In the following proposition we assume that the Hasse diagram of $(W, \preccurlyeq)$ is a Cayley graph of W, *i.e.*, its edges are labelled by the appropriate generators.

Lemma 4.3: *Let* W *be a finite Coxeter group. For* g, h *in* W*, let us write* $g \sim h$ *if* $\alpha_X(g)$ *equals* $\alpha_X(h)$*. This relation* $\sim$ *is a congruence of the semilattice* $(W, \preccurlyeq)$ *and the semilattice* $(W, \preccurlyeq)/_\sim$ *is isomorphic to* $(W_X, \preccurlyeq)$*. All congruence classes are isomorphic lattices and, moreover, their Hasse diagrams are isomorphic labelled graphs.*

PROOF: We know that α_X is a homomorphism from $(W, \preccurlyeq)$ onto $(W_X, \preccurlyeq)$. Hence $\sim$ is congruence and $(W, \preccurlyeq)/_\sim$ is isomorphic to $(W_X, \preccurlyeq)$. We only need to show that all the congruence classes are isomorphic.

If we have $a \neq b$ in W_X then there exists a natural bijection between the class containing a and the class containing b. This bijection sends ag to bg for each $g \in W^X$. But this bijection preserves the order $\preccurlyeq$: for all $g, h \in W^X$ we have $ag \preccurlyeq ah \Leftrightarrow g \preccurlyeq h \Leftrightarrow bg \preccurlyeq bh$. Therefore it is a semilattice isomorphism. It is evident that this isomorphism preserves also the labelling of edges: the edge from ag to agx is labelled by the generator x and so is the edge from bg to bgx. ■

All needed preliminary results are now at hand and we can prove the main result. Applying Proposition 3.2, together with Lemmas 4.2 and 4.3, we obtain:

Theorem 4.4: *Let* W *be a Coxeter group with the presentation* $\langle S; R \rangle$*. Let* X *be a subset of* S*. Then the semilattice* $(W, \wedge)$ *is isomorphic to a semidirect product of the semilattices* $(W_X, \wedge)$ *and* $(W^X, \wedge)$*.*

In the case of finite Coxeter groups, the ordered set $(W, \preceq)$ is a lattice. We would like to prove that this lattice is a semidirect product of lattices. To this aim, we need to know that α_X is compatible with the join.

Lemma 4.5: *Let g, h be in W such that $g \vee h$ exists. Then $\alpha_X(g) \vee \alpha_X(h)$ exists and we have $\alpha_X(g) \vee \alpha_X(h) = \alpha_X(g \vee h)$.*

PROOF: We have $\alpha_X(g) \preccurlyeq \alpha_X(g \vee h)$ and $\alpha_X(h) \preccurlyeq \alpha_X(g \vee h)$ hence the join has to exist. Moreover, we have $\alpha_X(g) \vee \alpha_X(h) \preccurlyeq \alpha_X(g \vee h)$. From $g \preccurlyeq g \vee h$ and the definition of ψ we obtain $\omega_X(g) \preccurlyeq \psi_{\alpha_X(g\vee h),\alpha_X(g)}(\omega_X(g \vee h))$. Now consider the element $j = \psi_{\alpha_X(g\vee h),\alpha_X(g)\vee\alpha_X(h)}(\omega_X(g\vee h))$. The previous observation along with Lemma 3.3 (iii) gives $\omega_X(g) \preccurlyeq \psi_{\alpha_X(g)\vee\alpha_X(h),\alpha_X(g)}(j)$ which implies $g \preccurlyeq (\alpha_X(g) \vee \alpha_X(h)j)$. The same can be said about h and therefore $\alpha_X(g) \vee \alpha_X(h)j$ is an upper bound of g and h and we have $\alpha_X(g \vee h) \preccurlyeq \alpha_X(g) \vee \alpha_X(h)$. ∎

Thus the mapping α is compatible with both lattice operations, forming a lattice endomorphism. Therefore we can apply Proposition 3.5 and obtain:

Theorem 4.6: *Let W be a finite Coxeter group with the presentation $\langle S; R\rangle$. Let X be a subset of S. Then the lattice $(W, \preccurlyeq)$ is isomorphic to a semidirect product of the lattices $(W_X, \preccurlyeq)$ and $(W^X, \preccurlyeq)$.*

We know at this point that the Cayley graphs are semidirect products but we do not know what products they exactly are, *i.e.*, we still have to describe the mapping ψ. As was observed after Lemma 3.3, it suffices to describe the covering triples $((k, h), (k', h'), s)$, *i.e.*, the triples satisfying $k'h' = khs$, where k' covers k in K. According to Lemma 4.2 (ii), we know that in this case we have $h = h'$.

Lemma 4.7: *Let g be in W^X and let s be in S such that gs is not X-reduced. Assume that a generator s' in S satisfies $g \preccurlyeq gs'$, the element gs' is X-reduced and $m_{s,s'}$ is finite. Then the element $g[s, s'\rangle^{m_{s,s'}}$ covers the element $g[s', s\rangle^{m_{s,s'}-1}$, and the element $g[s', s\rangle^{m_{s,s'}-1}$ is X-reduced.*

PROOF: We know that gs is not X-reduced and, according to Lemma 4.1, the element $t = \alpha_X(gs)$ is in S. Now Lemma 2.5 gives $(gs) \vee (gs') = g[s, s'\rangle^{m_{s,s'}}$. Since α_X is compatible with the join, we find $\alpha_X(g[s, s'\rangle^{m_{s,s'}}) = t$. Necessarily $\alpha_X(g[s', s\rangle^{m_{s,s'}-1}) \preccurlyeq t$ holds. If this relation were an equality, the element $(g[s', s\rangle^{m_{s,s'}-1}) \wedge (gs)$, which is g, would not be X-reduced. Hence $\alpha_X(g[s', s\rangle^{m_{s,s'}-1})$ is strictly smaller than t and this means that $g[s', s\rangle^{m_{s,s'}-1}$ is X-reduced. ∎

Lemma 4.8: *Let $g \neq 1$ be in W^X and assume that s is in S and gs is not X-reduced. Then there exist $g' \preccurlyeq g$ and s' in S such that $m_{s,s'}$ is finite, $g'[ss'\rangle^{m_{s,s'}} = gs$ holds and either the element $g's$ or the element $g's'$ is X-reduced.*

PROOF: There exists an s' in S such that $gs' \preccurlyeq g$ holds. According to Lemma 4.1 there exist a t from S satisfying $tg = gs$. Hence $gss' = tgs' \preccurlyeq tg = gs$ holds. Let us denote $g' = g \wedge gss'$. According to Lemma 2.5 we have $g' = gss \wedge gss' = gs[s, s'\rangle^{m_{s,s'}} = g[s', s\rangle^{m_{s,s'}-1}$. The elements g' and $g[s', s\rangle^{m_{s,s'}-2}$ are X-reduced, according to Lemma 2.8 (v). Now, the element $g's$ is one of the elements $g[s', s\rangle^{m_{s,s'}-2}, g[s, s'\rangle^{m_{s,s'}}$ and the element $g's'$ is then the other one. Since we have $gs = g'[s, s'\rangle^{m_{s,s'}} = g's \vee g's'$, necessarily $g[s, s'\rangle^{m_{s,s'}}$ is not X-reduced. ∎

Proposition 4.9: *Let W be a Coxeter group with presentation $\langle S; R\rangle$. Let X be a subset of S. Let k be in W_X and assume that s in X satisfies $k \preccurlyeq ks$. Let $h \neq 1$ be in W^X. Then ksh covers kh if and only if there exist $g \preccurlyeq h$ and s', s'' in S such that $m_{s',s''}$ is finite, and we have $kg[s', s''\rangle^{m_{s',s''}-1} = kh$ and $\alpha_X(kgs'') = ks$. Moreover, we have $ksh = khs'$, for m odd, and $ksh = khs''$, for m even.*

Proof: Nothing changes if we multiply all the elements by k^{-1} on the left. Then the equivalence is formulated as: sh covers h if and only if there exist $g \preccurlyeq h$ and s', s'' in S such that $m_{s',s''}$ is finite, $g[s', s''\rangle^{m_{s',s''}-1} = h$ holds and $\alpha_X(gs'')$ is equal to s. But '$\Leftarrow$' direction follows directly from Lemma 4.7 and '$\Rightarrow$' direction follows from Lemmas 4.2 (*ii*) and 4.8. Hence we need to establish only the equality $sh = hs'$ (resp. $sh = hs''$). Assume that m is odd. We know that $\alpha_X(gs'')$ is s. Hence there exists g' in W^X such that sg' covers g. According to Lemma 4.2 we have $g' = g$. Now we find

$$\begin{aligned} sh = sg[s', s''\rangle^{m_{s',s''}-1} &= gs''[s', s''\rangle^{m_{s',s''}-1} = \\ &= g[s', s''\rangle^{m_{s',s''}} = g[s', s''\rangle^{m_{s',s''}-1}s' = hs'. \end{aligned}$$

The case of m even is similar. ∎

We denote by $T_{k,k'}$ the set of all covering triples for a pair k, k' in W_X, that means the set of all covering triples $((k,h),(k',h),s')$, for some h in W^X and s' in S, satisfying $khs' = k'h$. We deduce from Proposition 4.9 an algorithmic method for inductively constructing the set $T_{k,k'}$. As as we observed after Lemma 3.3, this set determines the mapping ψ of the considered semidirect product completely.

Algorithm 4.10: Let W be a Coxeter group, with S the set of Coxeter generators, and let X be a proper subset of S. Fix a linear ordering $\leqslant$ on W^X that extends $\preccurlyeq$. Let $k \preccurlyeq k'$ be two elements in W_X, with $k' = ks$ for some s in S. We construct a set T of triples of the form $((k,h),(k',h),s')$, with h in W^X and s' in S, as follows. We initialize T to $\{((k,1),(k',1),s)\}$. We enter the induction on the length of h in W^X, starting with $h = 1$. Assume $((k,h),(k',h),s') \in T$. We successively consider all elements s'' in S satisfying hs'' in W^X and $m_{s',s''} < \infty$: for each of them, we add to T the triple

$$((k, h[s'', s'\rangle^{m_{s',s''}-1}),\ (k', h[s'', s'\rangle^{m_{s',s''}-1}),\ s'), \tag{4}$$

if $m_{s',s''}$ is even, or the triple

$$((k, h[s'', s'\rangle^{m_{s',s''}-1}),\ (k', h[s'', s'\rangle^{m_{s',s''}-1}),\ s''), \tag{5}$$

if $m_{s',s''}$ is odd. Then we go to the next h in the sense of $\leqslant$.

Lemma 4.11: *Algorithm 4.10 computes the set $T_{k,k'}$.*

Proof: We first show that each element of T belongs to $T_{k,k'}$. We have $ks = k'$ and therefore $((k,1),(k',1),s)$ is a covering triple. Next, whenever $((k,h),(k',h),s')$ is a covering triple then, according to Proposition 4.9, the construction (4) or (5) yields a covering triple.

On the other hand, suppose that there exists a triple $((k,h),(k',h),s')$ in $T_{k,k'}$ not belonging to T. Assume that h is $\leqslant$-minimal with this property. Each interval for $\preccurlyeq$ is finite, and therefore so is each interval for $\leqslant$. Hence such a minimal h exists. Proposition 4.9 guarantees the existence of an s'' in S and $g \preccurlyeq h$ in W^X verifying $g\langle s',s''\rbrack^{m_{s',s''}-1} = h$ and $k'g = kgt$, where t is either s' or s''. The triple $((k,g),(k',g),t)$ is a covering triple, and we have $g < h$. Therefore it belongs to $T_{k,k'}$. But then the triple $((k,h),(k',h),s')$ is reached from the triple $((k,g),(k',g),t)$ using (4) or (5), and therefore it belongs to $T_{k,k'}$, making a contradiction. ∎

Proposition 4.12: *Let W be a Coxeter group with the presentation $\langle S; R\rangle$. Let X be a subset of S. Then the infimum operation in $(W,\preccurlyeq)$ can be computed from the infimum operations in the semilattices W_X and W^X.*

Proof: The definition (3) of the meet in the semidirect product is:

$$(k,h)\wedge(k',h') = (k\wedge k', \psi_{k,k'}(h)\wedge\psi_{k',k}(h')).$$

We know how to compute the meet in the sublattices, so we need the mapping ψ only. This mapping is computed, according to Lemma 3.3, as follows:
- for $k' \preccurlyeq k$, this mapping is the identity mapping;
- for k, k' incomparable, we have $\psi_{k',k} = \psi_{k',k\wedge k'}$;
- for $k \prec k'$, we find a reduced decomposition of the element $k^{-1}k'$ of the form $s_1 s_2 \cdots s_k$ and compute:

$$\psi_{k',k} = \psi_{s_1,1}\circ\psi_{s_1 s_2,s_1}\circ\psi_{s_1 s_2 s_3,s_1 s_2}\circ\cdots\circ\psi_{s_1 s_2\cdots s_k,s_1 s_2\cdots s_{k-1}}; \qquad (6)$$

- for k' covering k, we have:

$$\psi_{k',k}(h) = \max\{h'\in W^X;\ (h'\preccurlyeq h) \text{ and } (((k,h'),(k',h'),s)\in T_{k,k'}),\ \text{for some } s\in S\}. \qquad (7)$$

The construction of the sets $T_{k,k'}$ is made by Algorithm 4.10. ∎

The same proposition can be also formulated for the supremum operation in the case of a finite Coxeter group. This operation is computed similarly.

The structure of the set $T_{k,k'}$ depends on the element $k^{-1}k'$, denoted s, only, not on the pair k,k' itself. More precisely, for h in W^X and s' in S, a triple $((k,h),(k',h),s')$ belongs to $T_{k,k'}$ if and only if the triple $((1,h),(s,h),s')$ belongs to the set $T_{1,s}$. Hence we deduce that the mapping $\psi_{k',k}$ is the same as the mapping $\psi_{k^{-1}k',1}$ and therefore it suffices to describe the mapping ψ (and the covering pairs) for the pairs $(s,1)$, with s in X.

Remark 4.13: In [10], the weak order lattice of a finite Coxeter group W is constructed using the mapping f from $W^X\times X$ to $S\cup\{\emptyset\}$ defined as follows: if ks covers k in W_X, with s in X, then, for each h in W^X, if $f(h,s)$ is empty, then ksh does not cover kh; otherwise we have $ksh = khf(h,s)$. This method is equivalent to our description using covering triples: if we have the set of covering triples, we can define $f(h,s)$ to be either s' if $((1,h),(s,h),s')$ is a covering triple or to be $\emptyset$ if sh does not cover h. The difference is that Le Conte de Poly-Barbut presents her method using a direct combinatorial approach and it is not clear how to compute the lattice operations $\wedge$ and $\vee$ from the mapping f.

5 APPLICATIONS

Let us consider the cases of the type B_n and the type $\tilde{A}_2$. We construct these Cayley graphs inductively using the method above.

Example 5.1: Consider the Coxeter group of type B_n, for $n \geqslant 2$, *i.e.*, the group with presentation

$$\begin{aligned} \langle s_1, s_2, \ldots, s_n; \qquad s_i^2 &= 1 && \text{for each } 1 \leqslant i \leqslant n, \\ (s_i s_j)^2 &= 1 && \text{for } |i-j| \geqslant 2, \\ (s_i s_{i+1})^3 &= 1 && \text{for } 2 \leqslant i < n, \\ (s_1 s_2)^4 &= 1 \rangle. \end{aligned}$$

We choose X to be $\{s_1, s_2, \ldots, s_{n-1}\}$. The set W^X is the set of all elements smaller than $s_n s_{n-1} \ldots s_3 s_2 s_1 s_2 s_3 \ldots s_{n-1} s_n$. Our aim is to construct the sets of covering triples T_{1,s_i}, for all $i < n$. The set W^X is already a chain (see [5]) and therefore we do not need to linearize the order $\preccurlyeq$ on W^X. We write b_i as a shortcut for $s_n s_{n-1} \cdots s_{i+1} s_i$ and c_i for $s_2 s_3 \cdots s_{i-1} s_i$.

Let us start with the element s_1: this element commutes with all elements s_j, for $2 < j \leqslant n$, and hence all triples $((1,b_j),(s_1,b_j),s_1)$ belong to T_{1,s_1}. Now we have $m_{s_1,s_2} = 4$ and hence we add the triple $((1,b_1 s_2),(s_1,b_1 s_2),s_1)$ to T_{1,s_1}. Now again s_1 commutes with all elements s_j, for $2 < j \leqslant n$, and hence all triples $((1,b_1 c_j),(s_1,b_1 c_j),s_1)$ belong to T_{1,s_1}. The result is shown in Figure 2.

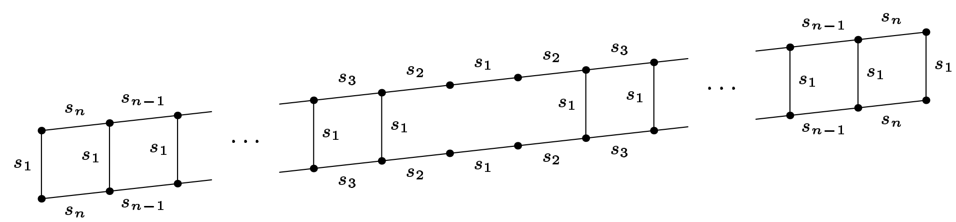

Figure 2: The mapping $\psi_{s_1,1}$ of the group of type B_n

Consider now an element s_i, with $i > 1$. This element commutes with all s_j, with $i+1 < j \leqslant n$. Hence the triples $((1,b_j),(s_i,b_j),s_i)$ belong to T_{1,s_i}. Now we have $m_{s_i,s_{i+1}} = 3$ and we add to T_{1,s_i} the triple $((1, b_i), (s_i, b_i), s_{i+1})$. The element s_{i+1} commutes with all s_j, for $j < i$ and therefore all triples $((1, b_j), (s_i, b_j), s_{i+1})$ and $((1, b_1c_j), (s_i, b_1c_j), s_{i+1})$ belong to T_{1,s_1}. Now we have again $m_{s_i,s_{i+1}} = 3$ and therefore we add the triple $((1, b_1c_{i+1}), (s_i, b_1c_{i+1}), s_i)$ to T_{1,s_i}. Finally the element s_i commutes with all s_j, for $j > i+1$ and we add $((1, b_1c_j), (s_i, b_1c_j), s_i)$ to T_{1,s_i}. The result is shown in Figure 3.

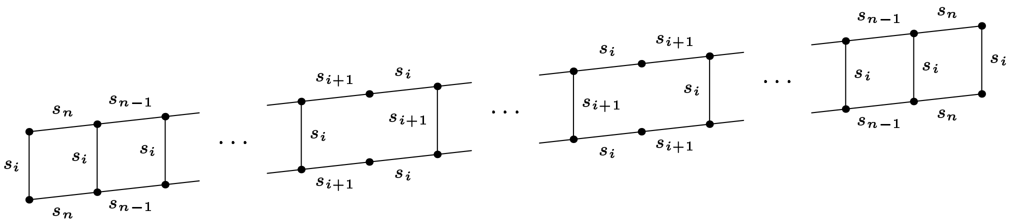

Figure 3: The mapping $\psi_{s_i,1}$, for $i \geqslant 2$, of the group of type B_n

So, the lattice of the group of type B_n is a semidirect product of the lattice associated with the group of type B_{n-1} and a $2n$-element linear lattice. The Cayley graph of the group of type B_n can be drawn followingly: we take the Cayley graph of the group of type B_{n-1} and we replace each edge by the corresponding "drawings" from pictures 2 and 3. As an example, we show the Cayley graph of the group of type B_3 in Figure 4.

Example 5.2: Let us consider the Coxeter group $\tilde{A}_2$, *i.e.*, the group with presentation

$$\begin{aligned} \langle s_1, s_2, s_3; \quad & s_i^2 = 1 && \text{for each } 1 \leqslant i \leqslant 3 \\ & (s_is_j)^3 = 1 && \text{for each } i \neq j \rangle. \end{aligned}$$

We choose X to be $\{s_1, s_2\}$. It is well-known [5] that the set W^X can be described as

$$W^X = \{\underbrace{s_3s_2s_1s_3s_2s_1\dots}_{m};\ m \geqslant 1\} \cup \{\underbrace{s_3s_1s_2s_3s_1s_2\dots}_{m};\ m \geqslant 1\} \cup \{g \in W;\ s_3s_1s_2s_1 \preccurlyeq g\}.$$

We compute the covering triples, according to Algorithm 4.10: As s_1 is an atom, the element $(s_1, 1)$ covers $(1, 1)$. Now the only generator s in S such that $1s$ is X-reduced is $s = s_3$. We have $m_{s_1,s_3} = 3$ and, applying Lemma 4.7, we deduce that $s_1s_3s_1$ covers s_3s_1. There are no other X-reduced elements of length 1 and hence we move forward to the triple $((s_1, s_3s_1), (1, s_3s_1), s_3)$. The element $s_3s_1s_2$ is the only successor of s_3s_1 which is X-reduced. We have $m_{s_2,s_3} = 3$ and this implies that $s_1s_3s_1s_2s_3$ covers $s_3s_1s_2s_3$. We continue in this way infinitely many times and obtain that the covering triples for the mapping $\psi_{s_1,1}$ are the triples (see Figure 5):

$$\begin{gathered} ((1, (s_3s_1s_2)^m), (s_1, (s_3s_1s_2)^m), s_1), \\ ((1, (s_3s_1s_2)^m s_3s_1), (s_1, (s_3s_1s_2)^m s_3s_1), s_3), \\ ((1, (s_3s_1s_2)^{m+1} s_3), (s_1, (s_3s_1s_2)^{m+1} s_3), s_2), \end{gathered}$$

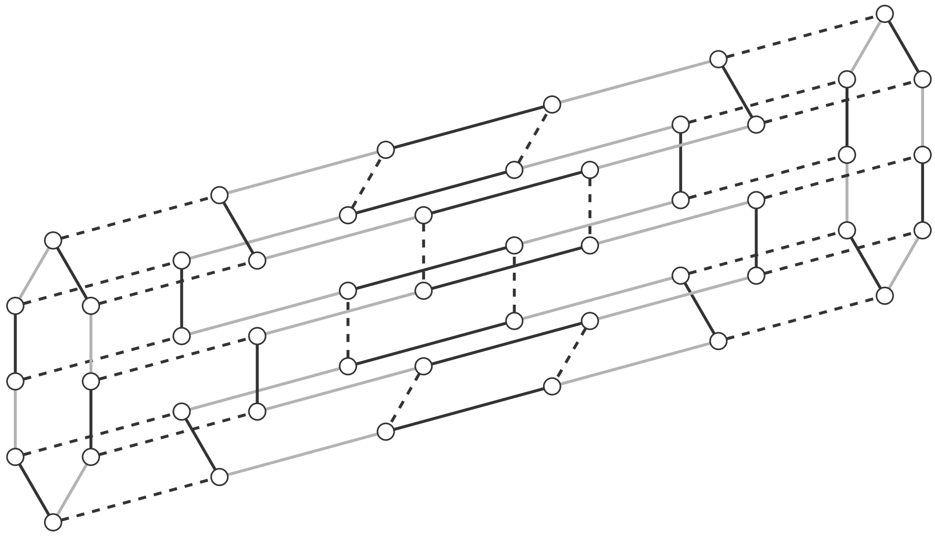

Figure 4: The Cayley graph (and also the weak order lattice) of the group of type B_3: the black edges represent s_1, the gray s_2 and the dashed s_3

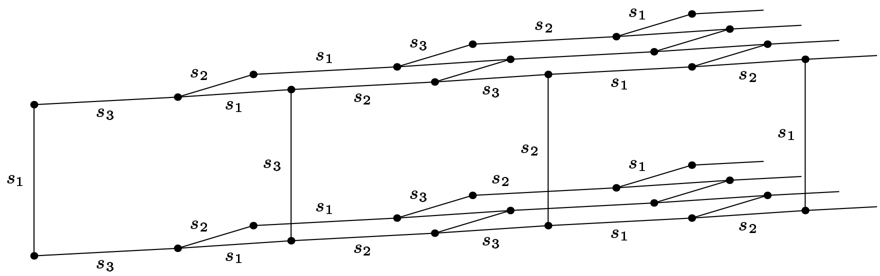

Figure 5: The mapping $\psi_{s_1,1}$ of the group of type $\tilde{A}_2$. The mapping $\psi_{s_2,1}$ is similar—it suffices to switch the labels s_1 and s_2 only.

with $m \geqslant 0$. Similarly, the covering triples for the mapping $\psi_{s_2,1}$ are

$$((1,(s_3s_2s_1)^m),(s_2,(s_3s_2s_1)^m),s_2),$$
$$((1,(s_3s_2s_1)^m s_3s_2),(s_2,(s_3s_2s_1)^m s_3s_2),s_3),$$
$$((1,(s_3s_2s_1)^{m+1}s_3),(s_2,(s_3s_2s_1)^{m+1}s_3),s_1).$$

with $m \geqslant 0$. The resulting picture is shown in Figure 6.

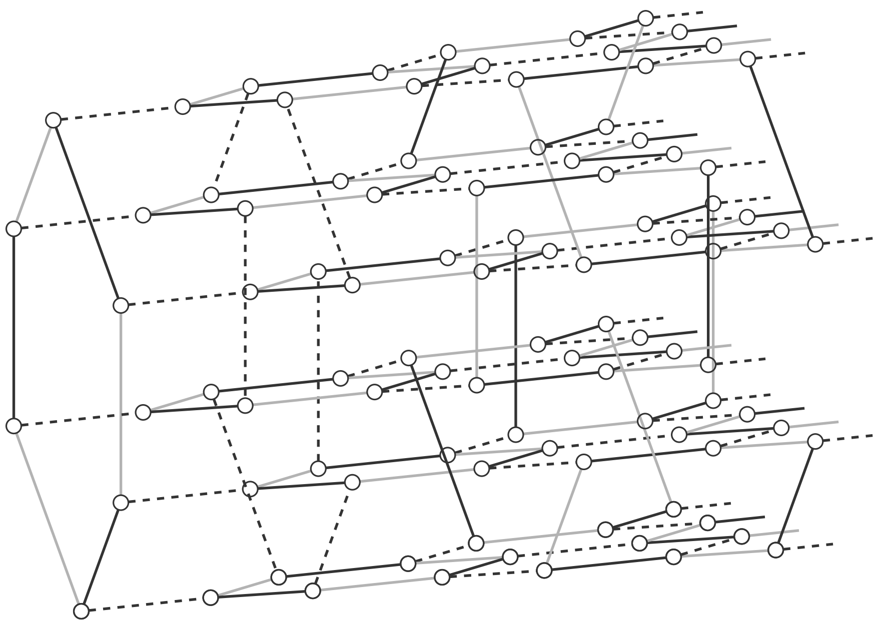

Figure 6: The Cayley graph (and also the weak order semilattice) of the group of type $\tilde{A}_2$: the black edges represent s_1, the gray s_2 and the dashed s_3

Let $g = s_1s_2s_3s_1s_2s_1s_3s_1$ and $g' = s_2s_3s_2s_1s_3s_2s_3s_1$, for instance. Let us compute the infimum of g and g' using Proposition 4.12. Using an algorithm of [10] we find $\alpha_X(g) = s_1s_2$ and $\alpha_X(g') = s_2$. These elements are incomparable and hence we have $\psi_{s_1s_2,s_2} = \psi_{s_1s_2,1}$ and $\psi_{s_2,s_1s_2} = \psi_{s_2,1}$. Now we compute $\psi_{s_1s_2,s_1}(s_3s_1s_2s_1s_3s_1)$ using (7). It consists of finding the greatest element h in W^X satisfying $h \preceq s_3s_1s_2s_1s_3s_1$ and $((1,h),(s_2,h),t)$ in T_{1,s_2}, for some $t \in \{s_1,s_2,s_3\}$. Such an element must be of the form $(s_3s_2)^a(s_1s_3)^b(s_2s_1)^c$, with $a \geqslant b \geqslant c \geqslant a-1$, and the greatest of them, but still smaller than $s_3s_1s_2s_1s_3s_1$, is s_3s_2. So we have $\psi_{s_1s_2,s_1}(s_3s_1s_2s_1s_3s_1) = s_3s_2$. Similarly we obtain $\psi_{s_1,1}(s_3s_2) = 1$ and therefore, using (6), we conclude $\psi_{s_1s_2,1}(s_3s_1s_2s_1s_3s_1) = 1$. In the same way we obtain $\psi_{s_2,1}(s_3s_2s_1s_3s_2s_3s_1) = s_3s_2s_1s_3$. Finally, equation (3) gives

$$g \wedge g' = (s_1s_2 \wedge s_2)(1 \wedge s_3s_2s_1s_3) = 1.$$

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References

[1] Birkhoff, G. Lattice theory, third edition; Amer. Math. Soc. Colloquium Publications 25, Amer. Math. Soc.: Providence, R.I., 1967

[2] Björner, A. Orderings of Coxeter groups. C. Greene (ed.), Cont. Math. 34, Amer. Math. Soc.: Providence, R.I., 1984, pp. 175–195

[3] Bourbaki, N. Groupes et algèbres de Lie, Chap. 4, 5 et 6"; Éléments de mathématiques, Fasc. XXXIV; Hermann: Paris, 1968.

[4] Deodhar, V. A splitting criterion for the Bruhat orderings on Coxeter groups. Comm. in Algebra 15 (9), 1987, 1889–1894

[5] Geck, M.; Pfeiffer G. Characters of Finite Coxeter Groups and Iwahori–Hecke Algebras; London Math. Soc. Monographs 21; Oxford Univ. Press, 2000.

[6] Humphreys, J.E. Reflection groups and Coxeter groups; Cambridge Univ. Press: Cambridge, 1976

[7] Jedlička, P. Prezentace s doplňky a levá distributivita (Czech). Diploma theses at the Charles university in Prague; Prague, 2001

[8] Jedlička, P. Semidirect products of lattices. preprint

[9] Le Conte de Poly-Barbut, C. Sur les treillis de Coxeter finis (French). Math. Inf. Sci. Hum. 125, 1994, 41–57

[10] Le Conte de Poly-Barbut, C. Treillis de Cayley des groupes de Coxeter finis. Constructions par récurrence et décompositions sur des quotients (French). Math. Inf. Sci. Hum. 140, 1997, 11–33