Semidirect products of lattices

PŘEMYSL JEDLIČKA

Abstract: The semidirect product of lattices is a lattice analogue of the semidirect product of groups. In this article we introduce this construction, show some basic facts and study a class of lattices closed under semidirect products. We also generalise this notion presenting the semidirect product of semilattices.

1 Introduction

The semidirect product of lattices was first introduced in [6]. This construction is inspired by the semidirect product in the group theory with the aim to have "the same" properties as its group counterpart: each lattice, constructed by the semidirect product, possesses a canonical congruence; this congruence is isoform, which means, all congruence classes are isomorphic sublattices. On the contrary, nearly all lattices with an isoform congruence can be expressed as semidirect products of smaller lattices.

The condition of the isoform congruence seems to be very limiting. However, for finite lattices we prove that given a lattice L with a congruence θ, we can embed L into a lattice L' extending θ to an isoform congruence θ'. A stronger result was proved by Grätzer, Quackenbush and Schmidt in [3] where they embedded any finite lattice into a lattice with all its congruences isoform, extending any congruence to an isoform one.

The notion of the semidirect product can be generalised for semilattices. Actually, the very first idea of the semidirect product appeared when studying lattices and semilattices of the weak order in a Coxeter group. The author described in [5] how to construct the (semi)lattice of the weak order of a Coxeter group using the semidirect product, starting with the weak order (semi)lattices of a parabolic subgroup and of the corresponding coset.

In this paper, some technical straightforward calculations are omitted. All the detailed calculations are written in the author's thesis [6].

Acknowledgement: The author wishes to thank Friedrich Wehrung, Patrick Dehornoy and the unknown referee for their help during the preparation of the text and their comments.

2 Semidirect product of lattices

We recall the definition of the semidirect product of groups: let $(G, \cdot)$ be such a group with a subgroup K and a normal subgroup H that we have $K \cap H = \emptyset$ and $KH = G$. In this case we can define such a mapping φ from K to $\mathrm{Aut}(H)$ that the product $K \times H$ with the operation $\bullet$, defined as

$$(k_1, h_1) \bullet (k_2, h_2) = (k_1 \cdot k_2,\ h_1 \cdot \varphi(k_1)(h_2)),$$

is a group isomorphic to G.

Let now L be such a lattice and let θ be a congruence of L that all congruence classes of θ are isomorphic. This congruence is called *isoform* (see [4]). We denote by H one of these congruence classes, and we assume that H has a least element 0_H and a greatest element 1_H.

We denote by K the factor lattice $L/_\theta$. There is a natural bijection between the sets L and $K\times H$: let a be in L and let $[a]_\theta$ be the congruence class of θ containing a. For each $[a]_\theta$, let $\eta_{[a]_\theta} = \eta_a$ be a fixed isomorphism from $[a]_\theta$ to H. Thus the mapping $a \mapsto ([a]_\theta, \eta_a(a))$ is the mentioned bijection from L to $K \times H$. Therefore we do not distinguish in the sequel between the sets L and $K \times H$ and we write the elements of L to be pairs (k, h) for k in K and h in H. And we also assume the set $K \times H$ to be equipped with an order—the image of the order of L.

Our goal is to build two binary operations, let us say $\sqcup$ and $\sqcap$, on the set $K \times H$ in order to achieve an isomorphism between $(L, \vee, \wedge)$ and $(K \times H, \sqcup, \sqcap)$. In the case of groups, we have the mapping φ there which enables us to reconstruct the operation $\cdot$ of the group G starting with the two subgroups. In the case of lattices, we have two operations and hence we need two mappings, let us call them φ and ψ.

The mappings φ and ψ from $K \times K$ to H^H are defined by the following equations; we recall that we do not distinguish the set L and the set $K \times H$ and therefore the order on L is naturally mapped to an order on $K \times H$.

$$(k,h) \vee (k \vee k', 0_H) = (k \vee k', \varphi(k,k')(h)), \tag{1}$$
$$(k,h) \wedge (k \wedge k', 1_H) = (k \wedge k', \psi(k,k')(h)). \tag{2}$$

Actually, we write $\varphi_{k,k'}(h)$ and $\psi_{k,k'}(h)$ rather than $\varphi(k,k')(h)$ and $\psi(k,k')(h)$, considering that for all k, k' in K, the mappings $\varphi_{k,k'}$ and $\psi_{k,k'}$ are mappings from H to H. You can also notice that in general there can be h' with $(k,h) < (k,h') \leqslant (k', \varphi_{k,k'}(h))$.

Lemma 2.1: *The mappings φ and ψ defined in (1) and (2) are well defined and they satisfy, for all k, k', k'' in K and h, h' in H, following conditions:*

$$\varphi_{k,k} = \psi_{k,k} = \mathrm{id}_H, \tag{3}$$
$$\varphi_{k,k' \vee k''} = \varphi_{k \vee k', k''} \circ \varphi_{k,k'}, \tag{4}$$
$$\psi_{k,k' \wedge k''} = \psi_{k \wedge k', k''} \circ \psi_{k,k'}, \tag{5}$$
$$h \leqslant \psi_{k,k \wedge k'} \circ \varphi_{k \wedge k', k}\ (h), \tag{6}$$
$$h \geqslant \varphi_{k,k \vee k'} \circ \psi_{k \vee k', k}\ (h), \tag{7}$$
$$\varphi_{k,k'}(h \vee h') = \varphi_{k,k'}(h) \vee \varphi_{k,k'}(h'), \tag{8}$$
$$\psi_{k,k'}(h \wedge h') = \psi_{k,k'}(h) \wedge \psi_{k,k'}(h'). \tag{9}$$

Proof: We prove only one half of the lemma since the other one follows from the duality. First let us notice that the element $(k,h) \vee (k \vee k', 0_H)$ is of the form $(k \vee k', h')$ for some h' in H.

Condition (3) follows from the definition.

Now we have to notice assuming (3) that Condition (4) is equivalent, for all k, k', k'' in K, to the following two conditions:

$$\varphi_{k,k'} = \varphi_{k,k \vee k'}, \tag{10}$$
$$\varphi_{k,k''} = \varphi_{k',k''} \circ \varphi_{k,k'} \qquad \text{for } k \leqslant k' \leqslant k''. \tag{11}$$

The implication (4) $\Rightarrow$ (10) and (11) is easy to see. So we want to prove the opposite direction. Let us have arbitrary k, k', k'' in K. Thus $k \leqslant k \vee k' \leqslant k \vee k' \vee k''$ implies $\varphi_{k, k \vee k' \vee k''} = \varphi_{k \vee k', k \vee k' \vee k''} \circ \varphi_{k, k \vee k'}$. Using Condition (10) we obtain Condition (4).

Hence, for proving (4) it is sufficient to prove (10) and (11). Condition (10) follows from the definition, so let us prove Condition (11). For $k \leqslant k' \leqslant k''$ in K, we have clearly

$(k, 0_H) \leqslant (k', 0_H) \leqslant (k'', 0_H)$. Therefore we can write, $(k'', \varphi_{k,k''}(h)) = (k, h) \vee (k'', 0_H) = (k, h) \vee (k', 0_H) \vee (k'', 0_H) = (k', \varphi_{k,k'}(h)) \vee (k'', 0_H) = (k'', \varphi_{k',k''} \circ \varphi_{k,k'}(h))$, for each h in H.

Let us study now Condition (6). Without any loss of generality we can suppose $k' \leqslant k$. We have

$$\begin{aligned}
(k', \psi_{k,k'} \circ \varphi_{k',k}(h)) &= (k, \varphi_{k',k}(h)) \wedge (k', 1_H) && \text{(by 2)}\\
&= ((k', h) \vee (k, 0_H)) \wedge (k', 1_H) && \text{(by 1)}\\
&\geqslant ((k', h) \wedge (k', 1_H)) \vee ((k, 0_H) \wedge (k', 1_H)) && \text{(dist.)}\\
&\geqslant (k', h) \wedge (k', 1_H) = (k', h)
\end{aligned}$$

Finally, a straightforward computation gives Condition (8). ■

It is useful to mention that we can prove similarly as in the proof, assuming (3), that Condition (5) can be expressed equivalently by conditions:

$$\psi_{k,k'} = \psi_{k,k\wedge k'}, \tag{12}$$

$$\psi_{k,k''} = \psi_{k',k''} \circ \psi_{k,k'} \qquad \text{for } k \geqslant k' \geqslant k''. \tag{13}$$

Example 2.2: The mapping φ is not compatible in general with the meet, *i.e.*, we do not have in general $\varphi_{k,k'}(h \wedge h) = \varphi_{k,k'}(h) \wedge \varphi_{k,k'}(h')$. We can see an example on Figure 1. We have $\varphi_{0_K,1_K}(a) = \varphi_{0_K,1_K}(b) = 1_H$ but the meet of a and b is 0_H and $\varphi_{0_K,1_K}(0_H)$ is 0_H.

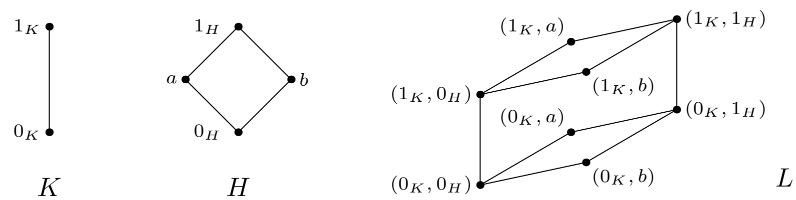


Figure 1: Example—the mapping φ is not compatible with $\wedge$

We have seen that the mappings φ and ψ satisfy some conditions. This means that these conditions are necessary. But they are also sufficient for reconstructing the lattice L.

Proposition 2.3: *Let K, H be two lattices, and let $\varphi, \psi : K \times K \to H^H$ be two mappings satisfying Conditions (3)–(9). Then the set $K \times H$. with binary operations $\sqcup$, $\sqcap$ defined as*

$$(k_1, h_1) \sqcup (k_2, h_2) = (k_1 \vee k_2,\, \varphi_{k_1,k_2}(h_1) \vee \varphi_{k_2,k_1}(h_2)), \tag{14}$$

$$(k_1, h_1) \sqcap (k_2, h_2) = (k_1 \wedge k_2,\, \psi_{k_1,k_2}(h_1) \wedge \psi_{k_2,k_1}(h_2)) \tag{15}$$

forms a lattice.

Proof: First we notice the following two properties for all k, k' in K and h, h' in H:

$$h \leqslant h' \Rightarrow \varphi_{k,k'}(h) \leqslant \varphi_{k,k'}(h'), \tag{16}$$

$$\varphi_{k,k'}(h \wedge h') \leqslant \varphi_{k,k'}(h) \wedge \varphi_{k,k'}(h'). \tag{17}$$

The first one follows from Condition (8) and the latter one from Condition (16). Analogically:

$$h \leqslant h' \Rightarrow \psi_{k,k'}(h) \leqslant \psi_{k,k'}(h'), \tag{18}$$

$$\psi_{k,k'}(h \vee h') \geqslant \psi_{k,k'}(h) \vee \psi_{k,k'}(h'). \tag{19}$$

A set equipped with two binary operations is a lattice, if they satisfy the laws of idempotency, commutativity, associativity and absorption. Idempotency holds here by Condition (3), commutativity follows from the definition. Now we prove associativity:

$$
\begin{aligned}
&\big((k_1,h_1)\sqcup(k_2,h_2)\big)\sqcup(k_3,h_3) & \\
&=\big(k_1\vee k_2\vee k_3,\ \varphi_{k_1\vee k_2,k_3}\big(\varphi_{k_1,k_2}(h_1)\vee\varphi_{k_2,k_1}(h_2)\big)\vee\varphi_{k_3,k_1\vee k_2}(h_3)\big) & \text{(by 14)}\\
&=\big(k_1\vee k_2\vee k_3,\ \varphi_{k_1\vee k_2,k_3}\circ\varphi_{k_1,k_2}(h_1)\vee\varphi_{k_1\vee k_2,k_3}\circ\varphi_{k_2,k_1}(h_2)\vee\varphi_{k_3,k_1\vee k_2}(h_3)\big) & \text{(by 8)}\\
&=\big(k_1\vee k_2\vee k_3,\ \varphi_{k_1,k_2\vee k_3}(h_1)\vee\varphi_{k_2\vee k_3,k_1}\circ\varphi_{k_2,k_3}(h_2)\vee\varphi_{k_3\vee k_2,k_1}\circ\varphi_{k_3,k_2}(h_3)\big) & \text{(by 4)}\\
&=\big(k_1\vee k_2\vee k_3,\ \varphi_{k_1,k_2\vee k_3}(h_1)\vee\varphi_{k_2\vee k_3,k_1}\big(\varphi_{k_2,k_3}(h_2)\vee\varphi_{k_3,k_2}(h_3)\big)\big) & \text{(by 8)}\\
&=(k_1,h_1)\sqcup\big((k_2,h_2)\sqcup(k_3,h_3)\big). & \text{(by 14)}
\end{aligned}
$$

For the absorption we need to prove

$$\varphi_{k_1\wedge k_2,k_1}(\psi_{k_1,k_2}(h_1)\wedge\psi_{k_2,k_1}(h_2))\leqslant h_1. \tag{$*$}$$

We obtain this by:

$$
\begin{aligned}
&\varphi_{k_1\wedge k_2,k_1}\big(\psi_{k_1,k_2}(h_1)\wedge\psi_{k_2,k_1}(h_2)\big) & \\
&\leqslant\varphi_{k_1\wedge k_2,k_1}\circ\psi_{k_1,k_2}(h_1)\wedge\varphi_{k_1\wedge k_2,k_1}\circ\psi_{k_2,k_1}(h_2) & \text{(by 17)}\\
&\leqslant\varphi_{k_1\wedge k_2,k_1}\circ\psi_{k_1,k_2}(h_1)=\varphi_{k_1\wedge k_2,k_1}\circ\psi_{k_1,k_1\wedge k_2}(h_1)\leqslant h_1. & \text{(by 12), (by 7)}
\end{aligned}
$$

And now the absorption:

$$
\begin{aligned}
&(k_1,h_1)\sqcup\big((k_1,h_1)\sqcap(k_2,h_2)\big) & \\
&=\big(k_1,\ \varphi_{k_1,k_1\wedge k_2}(h_1)\vee\varphi_{k_1\wedge k_2,k_1}\big(\psi_{k_1,k_2}(h_1)\wedge\psi_{k_2,k_1}(h_2)\big)\big) & \text{(by 15), (by 14)}\\
&=\big(k_1,\ h_1\vee\varphi_{k_1\wedge k_2,k_1}\big(\psi_{k_1,k_2}(h_1)\wedge\psi_{k_2,k_1}(h_2)\big)\big)=(k_1,h_1) & \text{(by 10), (by 3), (by $*$)}
\end{aligned}
$$

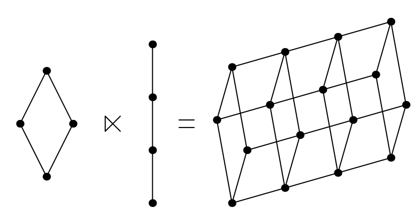
Dual associativity and absorption laws could be proven similarly. ∎

Definition 2.4: The lattice constructed in Proposition 2.3 is called a *semidirect product* of lattices and is denoted $K\ltimes_\psi^\varphi H$.

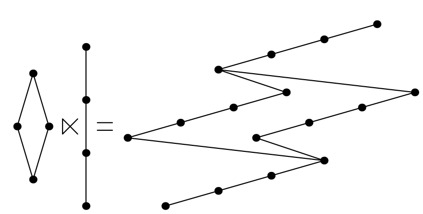
Proposition 2.5: *Let K,H,φ,ψ be as in Proposition 2.3. Than there exist a congruence θ on the semidirect product $K\ltimes_\psi^\varphi H$ which has K as the factor lattice and each congruence class is isomorphic to H.*

Proof: The considered congruence θ is the following one: $((k,h),(k',h'))\in\theta$ is equivalent to $k=k'$. ∎

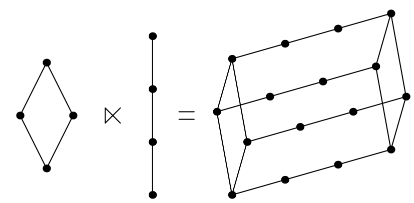
Example 2.6: Let K,H be arbitrary, let $\varphi_{k,k'}=\psi_{k,k'}=\mathrm{id}_H$ for all $k,k'\in K$. Then we can see immediately from Definitions (14) and (15) that we get $K\ltimes_\psi^\varphi H=K\times H$.



Example 2.7: Let K,H be arbitrary, for all $k\leqslant k'$, $h\in H$ let $\varphi_{k,k'}(h)=0_H$, $\psi_{k',k}(h)=1_H$. Then we have $(k,h)\leqslant(k',h')$ if and only if we have $k<k'$ in K or $k=k'$ and $h\leqslant h'$ in H. Therefore the semidirect product consists of $|K|$ copies of the lattice H arranged in the form of the lattice K.



Example 2.8: Let K, H be arbitrary, for all $k \leqslant k'$ let $\varphi_{k,k'}(h) = 1_H$ whenever $h > 0_H$; $\psi_{k',k}(h) = 0_H$, whenever $h < 1_H$. In this case, if K has at least two elements and H at least three elements, we always get a nonmodular lattice.



As we have seen, the applications φ and ψ defined in (1) and (2) define a lattice on $K \times H$. However we still need to prove that this lattice is isomorphic to the lattice L. In Lemma 2.1 we supposed that the lattice H possesses a least and a greatest elements. This is not much limiting in the lattice theory but we can still require weaker conditions. Recall that there exists a bijection between the lattices L and $K \times H$ and therefore the latter can be naturally equipped with an order associated with the order of L.

Proposition 2.9: *Let L be a lattice with an isoform congruence θ. Let K be the factor lattice $L/_\theta$ and let H be one of the congruence classes. If the following two conditions are fulfilled:*

$$\text{for each } h \in H,\ \text{the set } E_h = \{h' \in H;\ \exists\, k, k' \in K \colon (k,h) \leqslant (k',h')\} \text{ is lower bounded} \tag{20}$$

$$\text{for each } h \in H,\ \text{the set } E^h = \{h' \in H;\ \exists\, k, k' \in K \colon (k,h) \geqslant (k',h')\} \text{ is upper bounded} \tag{21}$$

then there exist mappings φ and ψ satisfying the conditions of the Proposition 2.3 and the lattice $K \ltimes_\psi^\varphi H$ is isomorphic to L.

Proof: First we denote, for each h in H, by $\underline{h}$ a lower bound of the set E_h and by $\overline{h}$ an upper bound of the set E^h. We can define the mappings similarly as we did it above:

$$(k \vee k', \varphi_{k,k'}(h)) = (k,h) \vee (k \vee k', \underline{h}), \tag{22}$$

$$(k \wedge k', \psi_{k,k'}(h)) = (k,h) \wedge (k \wedge k', \overline{h}). \tag{23}$$

However we have to prove that the definitions do not depend on the choice of the bounds. But that is true because we have $\varphi_{k,k'}(h) = \min\{h';\ (k \vee k', h') \geqslant (k,h)\}$ and $\psi_{k,k'}(h) = \max\{h';\ (k \wedge k', h') \leqslant (k,h)\}$. Now the proof does not differ much from the proof of Lemma 2.1 and it is thus left to the reader. ■

Since we have proven that the operation $\sqcup$, resp. $\sqcap$ are identical with the operation $\vee$, resp. $\wedge$, we use no more the symbols $\sqcup$, $\sqcap$ and keep using the standard lattice symbols.

Example 2.10: Conditions (20) and (21) cannot be avoided. Consider, for instance, the set $L = \{0,1\} \times \mathbb{Z}$ with the lexicographic order. It is a totally ordered set and hence a lattice which has an evident isoform congruence: the factor is $\{0,1\}$ and both the classes are isomorphic to $\mathbb{Z}$. However, no choice of φ would satisfy Condition (14): for $a < \varphi_{0,1}(b)$ we have $(1,a) = (0,b) \vee (1,a) = (1, \varphi_{0,1}(b) \vee a) = (1, \varphi_{0,1}(b))$, giving a contradiction.

3 Relations between φ and ψ

The semidirect product construction uses two mappings, φ and ψ, to equip the set $K \times H$ with a lattice structure. Nevertheless, the lattice order depends on one operation only which is, according to Condition (14), defined by

$$(k,h) \leqslant (k',h') \iff (k \leqslant k') \text{ and } (\varphi_{k,k'}(h) \leqslant h'). \tag{24}$$

Hence, one of those mappings has to be sufficient to describe the lattice structure. Therefore, in this section we describe the relation between φ and ψ. This relation is similar to the Galois correspondence or the adjoint mappings, with the difference that these notions are defined for complete lattices only.

At the beginning of this section (until Lemma 3.4) we suppose the following situation: we have a lattice $L = K \ltimes_{\psi}^{\varphi} H$ and we have fixed $k_1 \leqslant k_2 \in K$. We denote $\xi = \varphi_{k_1,k_2}$ and $\omega = \psi_{k_2,k_1}$. We are going to investigate what the relations between the mappings ξ and ω are.

Lemma 3.1: *Let us define mappings $\Xi : H \to H$; $\Xi = \omega \circ \xi$ and $\Omega : H \to H$; $\Omega = \xi \circ \omega$. Then we have: $\Xi(h) = \max\{\, j \in H;\ \xi(j) = \xi(h)\,\}$ and $\Omega(h) = \min\{\, j \in H; \omega(j) = \omega(h)\,\}$.*

PROOF: From the inequality (6) we obtain $h \leqslant \Xi(h)$. Then we use Condition (16) obtaining $\xi(h) \leqslant \xi(\Xi(h))$. And from (7) we gain $\xi(h) \geqslant \xi \circ \omega(\xi(h)) = \xi(\Xi(h))$. Hence, for all $h \in H$, we have $\xi(\Xi(h)) = \xi(h)$ and $h \leqslant \Xi(h)$.

The properties of Ω would be proven analogically. ■

We obtain immediately from the proof of the previous lemma:

Corollary 3.2: *$\xi\omega\xi = \xi$ and $\omega\xi\omega = \omega$.* ■

Lemma 3.3: *The mapping $\xi|_{\mathrm{Im}\,\omega}$ is a set bijection between Im ω and Im ξ. The mapping $\omega|_{\mathrm{Im}\,\xi}$ is the inverse bijection to $\xi|_{\mathrm{Im}\,\omega}$.*

PROOF: Let h be in Im ω. Then there exists h' in H satisfying $\omega(h') = h$. But then we have $\omega\xi(h) = \omega\xi\omega(h') = \omega(h') = h$. Hence $\omega\xi = \mathrm{id}_{\mathrm{Im}\,\omega}$ holds. Analogically we have $\xi\omega = \mathrm{id}_{\mathrm{Im}\,\xi}$. ■

Lemma 3.4: *The mapping ω is uniquely defined by the mapping ξ.*

PROOF: The reader himself can check that $\omega(h) = \max\{j \in H;\ \xi(j) \leqslant h\}$. ■

Using this result we can conclude generally:

Proposition 3.5: *Let K, H, φ, ψ be as in Proposition 2.3. Then the mapping ψ is uniquely defined by the mapping φ.*

PROOF: From Condition (12) we know that if k is incomparable with k' then $\psi_{k',k} = \psi_{k',k\wedge k'}$ holds. Next, $k \geqslant k'$ implies $\psi_{k',k} = id_H$ and the case $k < k'$ is solved in Lemma 3.4. ■

Once we know that one mapping is determined by the other one, we can omit one mapping from the definition of the semidirect product. We recall that we denote by $(h\,]$ the principal ideal generated by h.

Proposition 3.6: *Let K, H be two lattices and let φ be a mapping from $K \times K$ to H^H satisfying $\varphi_{k,k} = \mathrm{id}_H$, for all k in K, and Conditions (4) and (8), and also the condition*

$$\text{the set } \varphi^{-1}_{k_1,k_2}\Big((h\,]\Big) \text{ has its greatest element.} \tag{25}$$

Let us define a mapping $\psi : K \times K \to H^H$ as follows:

$$\psi_{k_1,k_2}(h) = \begin{cases} h; & \text{for } k_1 \leqslant k_2,\\ \max\big(\varphi^{-1}_{k_2,k_1}((h\,])\big); & \text{for } k_1 > k_2,\\ \psi(k_1, k_1 \wedge k_2)(h); & \text{otherwise.}\end{cases} \tag{26}$$

Then the set $K \times H$ equipped with operations $\sqcup, \sqcap$ defined in (14) and (15) forms a lattice.

Proof: The whole calculation is quite straightforward. The only important thing to notice is that the set $\varphi^{-1}_{k_1,k_2}(h\,]$ is always an ideal, due to (16). ∎

Now we can define semidirect products of lattices using only one mapping. Hence we denote the semidirect product $K \ltimes^{\varphi} H$ (respectively $K \ltimes_{\psi} H$) if the mentioned mapping is that one which describes the join (respectively the meet).

As we mentioned in the proof of Proposition 3.6, the set $\varphi_{k_1,k_2}(h\,]$ is an ideal and hence Condition (25) is trivial in finite lattices.

4 Semidirect product of semilattices

We have just seen that the semidirect product of lattices can be described using only one mapping. The same idea can be used for constructing semidirect products of semilattices.

Proposition 4.1: (i) *Let K, H be two join-semilattices and let φ be a mapping from $K \times K$ to H^H satisfying $\varphi_{k,k} = \mathrm{id}_H$, for all k in K and also Conditions (4) and (8). Then the set $K \times H$ equipped with the operation $\sqcup$ defined in (14) is a join-semilattice.*
(ii) *In addition, if K, H are complete ones and if the mapping φ verifies Condition (25) then $(K \times H, \sqcup)$ is a complete join-semilattice.*

Proof: The proof of part (i) is the same as the one of Proposition 2.3. So let us begin with the part (ii). Let us denote 0_H and 1_H the least and the greatest element of the semilattice H.

Since, for all k_1, k_2 in K, the set $\varphi^{-1}_{k_1,k_2}(0_H)$ is an ideal, we have $\varphi_{k_1,k_2}(0_H) \leqslant 0_H$, and therefore $\varphi_{k_1,k_2}(0_H) = 0_H$ holds. Hence, for all $k_1 \leqslant k_2$ and $h \in H$, we have $(k_1, 0_H) \leqslant (k_2, h)$. This gives us a remark, that $(0_K, 0_H)$ is the least element of $(K \times H, \sqcup)$.

Now consider a nonempty set M, a subset of $K \times H$. Denote $A = \{\, k \in K;\ \exists h \in H:\ (k,h) \in M \,\}$ and $a = \sup(A)$. Denote also $B = \{\, j \in H;\ \forall (k,h) \in M:\ (a,j) \geqslant (k,h) \,\}$. The set B is nonempty, because $(a, 1_H)$ is surely greater than or equal to each element of M. Hence we can define b to be $\inf(B)$. Evidently $b \in B$.

Now, knowing that (a,b) is an upper bound of M, we want to prove $(a,b) \leqslant (k,h)$, for (k,h), another upper bound of M. Clearly $a \leqslant k$; the case $a = k$ is evident. Suppose thus $a < k$. According to (25), the set $\varphi^{-1}_{a,k}((h\,])$ has its greatest element, let us say m. For each (l,j) in M we have $h \geqslant \varphi_{l,k}(j) = \varphi_{a,k} \circ \varphi_{l,a}(j)$ giving $\varphi_{l,a}(j) \leqslant m$. Therefore we have $m \in B$ and $b \leqslant m$, giving $(a,b) \leqslant (a,m) \leqslant (k,h)$. ∎

Corollary 4.2: *A semidirect product of complete lattices is a complete lattice.*

Proof: Every complete semilattice is already a complete lattice. ∎

We have constructed the semidirect product of semilattices, we denote it the same way as the semidirect product of lattices, that means $K \ltimes^{\varphi} H$ or $K \ltimes_{\psi} H$. There can be no confusion, since lattices, as well as semilattices, are nothing but partially ordered sets. Hence the semidirect product is also a partially ordered set, its properties depend only on the sets K and H and the considered mapping.

Example 4.3: Condition (25) is needed in order to achieve a lattice: consider the lattice H to be the closed real interval $[0,2]$, the lattice K to be $\{0,1\}$ and $\varphi_{0,1}$ to be the mapping $x \mapsto \lfloor x \rfloor$. Both lattices are complete however the semidirect product $K \ltimes^{\varphi} H$ is a semilattice only—the set $\{(1,0),(0,1)\}$ has not an infimum.

Proposition 4.4: *Let L be a join-semilattice and let θ be an isoform congruence of L. Let us denote $K = L/_\theta$ and by H one of the congruence classes. If L verifies Condition (20) then there exists such a mapping φ from $K \times K$ to* $\mathrm{End}(H)$ *that L is isomorphic to $K \ltimes^\varphi H$.*

Proof: We need to prove that the conditions of Proposition 4.1 are fulfilled. However the proof is just a copy of the one of Proposition 2.9. ■

5 Arbitrary congruence

In this section we show how to embed an arbitrary finite lattice with an arbitrary nontrivial congruence into a semidirect product, namely a semidirect product of the quotient and classes of the congruence.

Definition 5.1: Let $L_1, L_2, \ldots, L_\kappa$ be lattices. We denote by $\sum_{i<\kappa} L_i$ the *ordinal sum* of lattices, defined to be the disjoint union $\biguplus_{i<\kappa} L_i$ equipped with the following order $\leqslant$:

$$a \leqslant b \text{ in } L \iff ((a, b \in L_i) \text{ and } (a \leqslant_{L_i} b)) \text{ or } ((a \in L_i, b \in L_j) \text{ and } (i < j)).$$

It is clear that this structure is a lattice again, that this operation is associative and that if we produce an ordinal sum of ordinal sums then it is the same as doing one big ordinal sum at once. We use ordinal sums for gluing together all classes of an arbitrary congruence.

Proposition 5.2: *Let L be a finite lattice and let θ be a nontrivial congruence on L. Then L embeds into a semidirect product of $L/_\theta$ and an ordinal sum made by congruence classes of θ and one-element lattices in such a way that this embedding extends θ into the canonical congruence of the semidirect product.*

Proof: We denote by K the lattice $L/_\theta$ and by $[a]_\theta$ the congruence class containing a in L. We denote α the homomorphism from L onto K, Let us choose now an ordering $\trianglelefteq$, a linear extension of the order $\leqslant_K$. (By $\triangleleft$ we denote the strict version of $\trianglelefteq$.) Using this order, we construct a lattice H to be an ordinal sum of congruence classes of θ':

$$H \stackrel{\text{def}}{=} \sum_{k \in K, \text{ ordered by } \trianglelefteq} \alpha^{-1}(k).$$

We denote by γ the natural mapping from H to K which sends each summand to its associated element k in K. Finally, we denote 0_k and 1_k the least and the greatest element of the class k.

Each summand of H is a copy of $\alpha^{-1}(k)$, for some k in K. Therefore there exists a natural bijection β_k between the sets $\alpha^{-1}(k)$ and $\gamma^{-1}(k)$. This mapping is clearly an isomorphism. If we unite all betas we obtain a bijection of the sets L and H: we define $\beta(a) = \beta_{\alpha(a)}(a)$.

We define now a mapping φ from $K \times K$ to H^H, which, as we will see, forms a semidirect product. We define $\varphi_{k,k'}$ for all $k <_K k'$ in K as follows (for illustration, see Example 5.3):

$$\varphi_{k,k'}(h) = \beta(0_{\gamma(h)}) \qquad \text{for } h \leqslant_H \beta(0_k) \text{ or } \beta(1_{k'}) <_H h \tag{27}$$
$$\varphi_{k,k'}(h) = \beta(1_{k'}) \qquad \text{for } \beta(1_k) \leqslant_H h \leqslant_H \beta(1_{k'}) \tag{28}$$
$$\varphi_{k,k'}(h) = \beta(\beta^{-1}(h) \vee_L 0_{k'}) \qquad \text{for } \beta(0_k) <_H h \leqslant_H \beta(1_k) \tag{29}$$

We have to check first that this mapping involves a semidirect product. We start to prove that in is compatible with the join, for all $k <_K k' \in K$, $\varphi_{k,k'}$.

We investigate whether the expression $\varphi_{k,k'}(h) \vee_H \varphi_{k,k'}(h')$ is equal to $\varphi_{k,k'}(h \vee_H h')$, for all h, h' in H. We assume $\gamma(h) \trianglelefteq \gamma(h')$, without loss of generality. Since we have $\gamma(\varphi_{k,k'}(h)) \trianglelefteq \gamma(\varphi_{k,k'}(h'))$, the claim has to be true for $\gamma(h) \triangleleft \gamma(h')$.

So, let us assume $\gamma(h) = \gamma(h')$. The case $\gamma(h) \neq k$ is evident hence suppose $\gamma(h) = \gamma(h') = k$. If one of h, h' is equal to 0_k or 1_k then it is clear. Hence suppose it is not so. Then we have

$$\begin{aligned}
\varphi_{k,k'}(h) \vee_H \varphi_{k,k'}(h') &= \beta_{k'}(\beta^{-1}_{\gamma(h)}(h) \vee_L 0_{k'}) \vee_H \beta_{k'}(\beta^{-1}_{\gamma(h')}(h') \vee_L 0_{k'}) && \text{(by 29)} \\
&= \beta_{k'}(\beta^{-1}_{\gamma(h)}(h) \vee_L \beta^{-1}_{\gamma(h')}(h') \vee_L 0_{k'}) \\
&= \beta_{k'}(\beta^{-1}_{\gamma(h)}(h \vee_H h') \vee_L 0_{k'}) = \varphi_{k,k'}(h \vee_H h'). && \text{(by 29)}
\end{aligned}$$

We used the facts that the mapping β is an isomorphism, when restricted to only one congruence class, and that we have $\gamma(h) = \gamma(h')$ and $\alpha(\beta^{-1}(h) \vee_L 0_{k'}) = k \vee_K k' = \alpha(\beta^{-1}(h') \vee_L 0_{k'})$.

Now we prove that $\varphi_{k,k''} = \varphi_{k',k''} \circ \varphi_{k,k'}$ holds, for all $k <_K k' <_K k''$ in K. Really, we have

$$\begin{aligned}
\varphi_{k',k''} \circ \varphi_{k,k'}(h) &= \beta(\beta^{-1}(\beta(\beta^{-1}(h) \vee_L 0_{k'})) \vee_L 0_{k''}) && \text{(by 29)} \\
&= \beta(\beta^{-1}(h) \vee_L 0_{k'} \vee_L 0_{k''}) = \varphi_{k,k''}(h), && \text{(by 29)}
\end{aligned}$$

for all $h \in H$ with $\beta(0_k) <_H h <_H \beta(1_k)$. But the other cases are simple.

We have proven all conditions needed for $\varphi_{k,k'}$, where $k < k' \in K$ and now it suffices to define $\varphi_{k,k} = \mathrm{id}_H$ and $\varphi_{k,k'} = \varphi_{k,k \vee k'}$, for all k, k' in K, and we have defined all the mapping φ. And we have also proven that this mapping involves the construction of the lattice $K \ltimes^{\varphi} H$.

Now let us consider the set $\tilde{L} = \{(k,h);\ k \in K,\ h \in H,\ \gamma(h) = k\}$. We claim that this set is a sublattice of $K \ltimes^{\varphi} H$ and it is isomorphic to L.

Let $(k,h), (k',h')$ be arbitrary elements of $\tilde{L}$. Then $(k,h) \vee_{K \ltimes^{\varphi} H} (k',h')$ is, by definition

$$\begin{aligned}
&(k \vee_K k', \varphi_{k,k'}(h) \vee_H \varphi_{k',k}(h')) \\
&= (k \vee_K k', \beta_{k \vee k'}(\beta^{-1}(h) \vee_{L'} 0_{k \vee k'}) \vee_H \beta_{k \vee k'}(\beta^{-1}(h') \vee_{L'} 0_{k \vee k'})) \\
&= (k \vee_K k', \beta_{k \vee k'}(\beta^{-1}(h) \vee_{L'} \beta^{-1}(h') \vee_{L'} 0_{k \vee k'})) \\
&= (k \vee_K k', \beta(\beta^{-1}(h) \vee_{L'} \beta^{-1}(h'))).
\end{aligned}$$

We can see that $\alpha(\beta^{-1}(h) \vee_{L'} \beta^{-1}(h'))$ is $k \vee_K k'$. Therefore $(k,h) \vee_{K \ltimes^{\varphi} H} (k',h')$ belongs to $\tilde{L}$.

For proving the same thing about the meet we need a similar definition of the dual mapping ψ as we have for the mapping φ. However, we have already proven that, according to Definition (26), for each $k, k' \in K$, $0_k \neq \beta^{-1}(h) \neq 1_k$ and $\gamma(h) = k$, we have $\psi_{k',k}(h) = \beta(\beta^{-1}(h) \wedge_{L'} 1_k)$. Hence the proof saying that $(k,h) \wedge_{K \ltimes^{\varphi} H} (k',h')$ remains in $\tilde{L}$ is the same as the one for the join.

The only thing left at the moment, is to prove $L \cong \tilde{L}$. We define a mapping $f : L \to \tilde{L}$ which sends a to $(\alpha(a), \beta(a))$. This mapping is clearly a bijection and it is a homomorphism since β is a bijection on congruence classes. It is hence an isomorphism and the lattice L embeds into $K \ltimes^{\varphi} H$. ■

Example 5.3: Here we show an example how the construction works in a specific case. Let L be as in Figure 2. We have chosen one nontrivial congruence and thick edges represent classes of this congruence. Then we construct the lattices K and H.

We construct the semidirect product, we have considered in Proposition 5.2 (see Figure 3). The lattice L is finite, hence we need not to embed it into another lattice L' first in order to guarantee that all congruence classes have its least and greatest elements. We can see that L embeds directly into the semidirect product $K \ltimes^{\varphi} H$ (the black vertices).

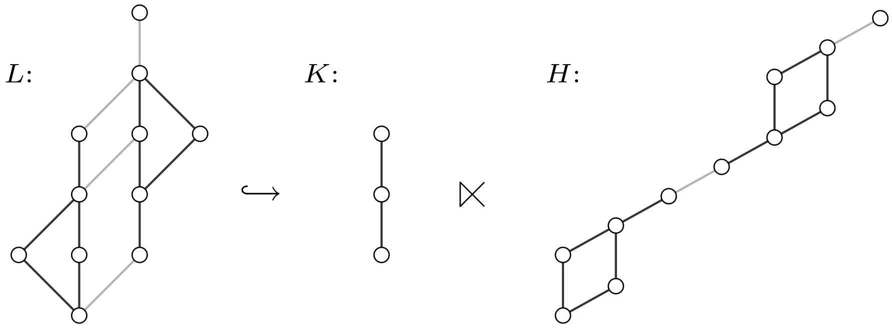


Figure 2: Lattices L, K and H

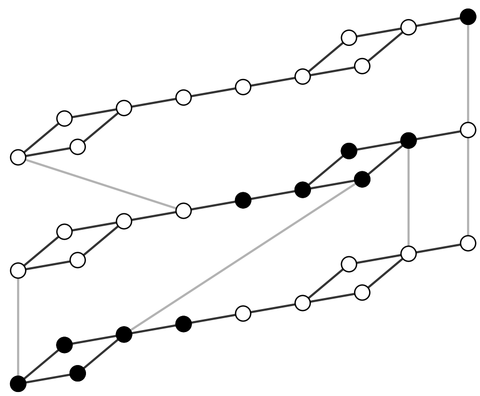


Figure 3: The lattice L embeds into $K \ltimes^{\varphi} H$

References

[1] G. Birkhoff: "Lattice theory", third edition; Amer. Math. Soc. Colloquium Publications 25, Amer. Math. Soc., Providence, R.I., 1967

[2] R. Freese, J. Ježek, J. B. Nation: "Free Lattices", Math. Surveys and Monographs 42, Amer. Math. Soc., Providence, R.I., 1995

[3] G. Grätzer, R. W. Quackenbush, E. T. Schmidt: *Congruence-preserving extensions of finite lattices to isoform lattices,* Acta Sci. Math. (Szeged) 70, 2004, no. 3-4, 473–494

[4] G. Grätzer, E. T. Schmidt: *Finite Lattices with Isoform Congruences,* Tatra Mountain Math. Publ. 27, 2003, 111–124

[5] P. Jedlička: *Combinatorial Construction of the Weak Order of a Coxeter Group*, Comm. in Alg. 33, no. 5, 2005, 1447–1460

[6] P. Jedlička: "Treillis, groupes de Coxeter et les systèmes LDI" (French and Czech), Ph.D. thesis, Université de Caen, 2004, Caen

[7] P. Jipsen, H. Rose: "Varieties of Lattices", Lecture Notes in Math., 1533, Springer-Verlag, 1992